

1956

# An iterative method applied to wave guide and resonant cavity problems

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197  
AN ITERATIVE METHOD APPLIED TO WAVE GUIDE  
AND RESONANT CAVITY PROBLEMS

by

Robert Grover Brown

A Dissertation Submitted to the  
Graduate Faculty in Partial Fulfillment of  
The Requirements for the Degree of  
DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

Approved:

Signature was redacted for privacy.

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1956

UMI Number: DP12413

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## I. INTRODUCTION

### A. Statement and Scope of Problem

The objective of this thesis is to present an iterative method for the solution of resonant cavity and wave guide problems involving boundaries of arbitrary shape. In the ensuing process of solving for the field configuration the eigenvalue associated with the particular mode of interest is also found. In both the cavity and wave guide cases the problems will be idealized by considering the dielectric material to be homogeneous and lossless and the surrounding metallic walls as having infinite conductivity. Also, only closed bounding surfaces will be considered in the cavity problem.

The resonant cavity and wave guide problems are considered together because of the similarity of the boundary value problems involved in both cases. The wave guide problem reduces to one of solving the scalar wave equation in two dimensions subject to certain boundary conditions. In the case of the resonant cavity the vector wave equation must be solved with its associated boundary conditions.

The iterative process to be used is essentially an extension of Picard's method of successive approximations. One starts by assuming a function which satisfies the boundary conditions of the problem and then uses it along with the differential equation to obtain a second function which will be closer to the desired solution than the original one. The

second one is then used to obtain a third and so forth until a solution within the desired accuracy is obtained. With each iteration one also obtains an approximate value for the eigenvalue associated with the solution. This approximation also improves with each step.

Although the idea of applying the method of successive approximations to boundary-value type problems is not new, the particular technique used here, which involves simultaneous successive approximations for both the eigenfunction and eigenvalue of the problem, is believed to be novel.

#### B. Review of Literature

The subject of wave guides and cavities is of course a classical one and literature concerning it is quite extensive. Nearly every text dealing with electromagnetic waves considers these problems to some extent or other. The usual procedure is to solve a few cases involving simple geometry and let it go at that. There are however some notable exceptions.

One of these is Slater's "Microwave Electronics" (12) in which the author approaches the subject from quite a general viewpoint. This is probably the most comprehensive treatment of the general theory of resonant cavities available in the form of a text. However, in spite of the general approach, there is no hint given as to how one might go about obtaining a numerical answer to a problem involving anything other than the standard elementary geometric configurations.

Stratton's "Electromagnetic Theory" (13) is also a famous text in this field and contains quite a bit of general theory regarding the problem.

Here again, though, there is no suggestion as to how to go about obtaining numerical answers to the general problem.

While mentioning books on this subject one would hardly dare overlook the Radiation Laboratory Series. Particularly pertinent to this subject are Vol. 10, Marcuvitz's "Wave Guide Handbook" (5) and Vol. 8, Montgomery's "Principles of Microwave Circuits" (6). In addition to this series of books there are numerous other Radiation Laboratory publications which are available to a limited extent. Two of these were particularly helpful. One was the two volume edition of "Notes on Microwaves" by Hansen (2) in which the cavity and wave guide problems are treated rather thoroughly. Also, a lucid treatment of the Ritz method as applied to these problems is given here. The other was Report 43-34, "The Theory of Obstacles in Resonant Cavities and Wave Guides" by Schwinger (11). This was the source of the idea of using a dyadic for a Green's function in the solution of the vector wave equation.

In addition to the above cited references which are of general interest, there are others of interest only with regard to certain phases of the problem. These will be referred to in the body of the thesis as the need arises.



## II. MATHEMATICAL BACKGROUND

### A. The Green's Function

The Green's function is useful in much of the following work, so a few paragraphs will be devoted to the subject at this point. This subject is discussed in detail in such standard differential equations texts as Ince (3) or Coddington and Levinson (1), and they may be referred to for a more thorough treatment of the subject. A little more restricted viewpoint, but perhaps more fitting for the problem at hand, will be given here.

Let  $G(x, \xi)$  denote the Green's function for the differential system

$$\begin{aligned} Ly &= 0, & a < x < b \\ U_i &= 0 \end{aligned} \quad (1)$$

where  $L$  is an  $n$ th order linear differential operator,  $y$  is a function of  $x$  and  $U_i$  represents the boundary condition equations involving linear combinations of  $y$  and its  $n-1$  derivatives evaluated at  $x=a$  and  $x=b$ . This is the same notation used in the above cited references.

Next consider the similar but nonhomogeneous differential system

$$\begin{aligned} Ly &= r(x) \\ U_i &= 0 \end{aligned} \quad (2)$$

where  $r(x)$  is arbitrary within the interval from  $a$  to  $b$ . One of the very useful properties of the Green's function for (1) is that it enables one to write the solution to the corresponding nonhomogeneous problem (2)

immediately as

$$y(x) = \int_a^b G(x, \xi) r(\xi) d\xi \quad (3)$$

As a matter of fact one might take the rather restricted viewpoint of using (3) as a means of defining the Green's function. This approach is useful in determining some of the essential properties of this function.

With this in mind let  $L$  operate on both sides of (3) and note that  $L$  operates with respect to  $x$  and not  $\xi$ . This along with (2) leads to

$$L y(x) = \int_a^b [L G(x, \xi)] r(\xi) d\xi = r(x). \quad (4)$$

If, for arbitrary  $r(x)$ , the above integral is to yield  $r(x)$ , then it can be seen that  $L G(x, \xi)$  must be a Dirac delta function, i.e.

$$L G(x, \xi) = \delta(x - \xi) \quad (5)$$

Thus  $G$  (as a function of  $x$ ) must satisfy the homogeneous differential equation (1) at every point within the interval except where  $x = \xi$ . At this point there must be an upward jump of unity in the  $n-1$  derivative of  $G$  if  $L G$  is to be a delta function at  $x = \xi$ .

Also, if  $y$  is to satisfy the boundary conditions as specified by  $U_i = 0$ , then  $G(x, \xi)$  must also. This can be verified by substituting the expression for  $y$  of (3) into the boundary condition equations and noting that  $r(\xi)$  factors out of each term within the integral leaving the  $U_i$  expression with  $G$  in the role of  $y$ . Thus, since the boundary condition expressions are homogeneous,  $G$  must satisfy them just as  $y$  does.

Thus, in summary it can be said that a Green's function must have the following properties:

1.  $G(x, \xi)$  as a function of  $x$  must satisfy the homogeneous differential equation  $LG = 0$  except at  $x = \xi$ .
2. At  $x = \xi$ , the  $n-1$  derivative of  $G(x, \xi)$  must have an upward jump of unity if  $LG$  is to be a delta function.
3.  $G(x, \xi)$  must satisfy the same boundary conditions as imposed on  $y$ .

If the above are true, then the solution of the nonhomogeneous problem (2) can be written by inspection as in (3).

#### B. Picard's Method of Successive Approximations

As mentioned previously the approach to the wave guide and cavity problems is essentially one of extending Picard's method of successive approximations to apply to these problems. So a brief resume of the method will be given. As in the case of the Green's function, a more detailed discussion will be found in Ince (3).

Perhaps the best way of introducing the method would be by means of an example. Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad (6)$$

with the boundary condition

$$y(a) = b. \quad (7)$$

Now successive approximations are formed by first assuming any arbitrary curve through the point  $(a, b)$  and denoting it  $y_0(x)$ . Then with  $y_0$  substituted into the right side of (6), the equation may be integrated yielding a new function which will also go through the point  $(a, b)$  if the constant of integration is chosen properly. Let this function be denoted by  $y_1(x)$ . The process can then be repeated using  $y_1$ , rather than  $y_0$  in the right side of (6) and a  $y_2$  obtained, and so forth, leading to the recursion expression

$$\frac{dy_n}{dx} = f(x, y_{n-1}) \quad (8)$$

$$y_n(a) = b. \quad (9)$$

It is known that this process will converge to the appropriate solution providing  $f(x, y)$  is sufficiently well behaved. Note that the method involves no guess work beyond choosing the initial  $y_0$ , and no matter how much this may differ from the correct solution, the process converges, with each integration leading to a function which is a little closer to the desired result than the previous one.

Higher order differential equations can be handled by writing only the highest order term on the left side of the equation and all others on the right. From here on the process is essentially the same as in the first order case just described.

The treatment of two-point boundary-value type problems by this method is somewhat more complicated and is discussed in some detail by Picard (9) and also to a lesser extent by Ince (3). As the technique to

be used here differs considerably from that of the references cited, there is no need to elaborate at this point.

### C. Application to the Sturm-Liouville Problem

Before going to the wave equation problems at hand, some insight into the more complex problem can be gained by first considering the corresponding wave equation problem in one dimension. This would be the differential system

$$\frac{d^2 y}{dx^2} + k^2 y = 0 \quad (10)$$

$$U_i = 0$$

where  $k^2$  is the parameter of the system. This is just a special case of a more general type of system known as the Sturm-Liouville system. When written in normal form<sup>a</sup> the general equation is

$$\frac{d^2 y}{dx^2} + [\lambda - q(x)] y = 0, \quad a \leq x \leq b \quad (11)$$

$$U_i = 0$$

where  $\lambda$  is a parameter of the system. In discussing some of the generalities to follow it is just as easy to deal with the general case (11) as the special one of (10), so this will be done.

---

<sup>a</sup>See Ince (3) p. 270 for the transformation leading to this form for the differential equation.

Generally speaking, solutions of (11) which will satisfy the boundary conditions exist only for discrete values of  $\lambda$ . These values are called the eigenvalues of the problem and corresponding solutions the eigenfunctions of the problem. A great deal of mathematical theory has been built up about such functions, and their properties are well known. Again such standard texts as Ince (3) or Coddington and Levinson (1) may be referred to for a detailed treatment of the subject.

In order to apply the method of successive approximations to this problem let (11) be written in the form

$$\begin{aligned} L y &= -\lambda y \\ U_i &= 0 \end{aligned} \tag{12}$$

where the operator  $L$  has been introduced as a matter of convenience and is defined as

$$L \equiv \frac{d^2}{dx^2} - f(x) . \tag{13}$$

Also, let  $G(x, \xi)$  denote the Green's function for the system

$$\begin{aligned} L y &= 0 \\ U_i &= 0 . \end{aligned} \tag{14}$$

The basic idea of the method is the same as before. An initial  $y^0$  is chosen which satisfies the boundary conditions. Then it is substituted into the right side of (12) and another function is generated by (12) which will be denoted  $y^1$ . Note that superscripts are being used rather than

subscripts as before. This should serve to avoid some confusion with subscripts later on, and where quantities are to be raised to a power they will be enclosed in brackets.

There is a slight difficulty that arises at this point, however. The parameter  $\lambda$  is not known and thus  $y'$  cannot be found from (12) until  $\lambda$  is specified. As it will be convenient throughout the iteration process to deal with normalized functions, each time the differential equation (12) is integrated to obtain a new approximation,  $\lambda$  will be chosen such as to normalize the new function. Thus a different value of  $\lambda$  will be associated with each successive approximation and these will be denoted with a superscript also. It is this feature of the method which differs from that given in Picard (9) and Ince (3). Thus the successive approximations are generated as follows:

$$\begin{aligned} L y' &= -\lambda^1 y^0 \\ L y^2 &= -\lambda^2 y' \\ &\dots \\ L y^n &= -\lambda^n y^{n-1} \end{aligned} \tag{15}$$

where  $\lambda^1, \lambda^2, \dots$  are chosen such as to normalize the resulting  $y', y^2, \dots$ .

The convergence of such an iterative process will be investigated next. It will be shown that as  $n$  tends to infinity,  $y^n$  approaches a solution of (12), and at the same time  $\lambda^n$  approaches the corresponding eigenvalue of the problem.

Consider the first step of the iterative process -- that of obtaining  $y'$ . From (15)  $y'$  can be written explicitly as

$$y'(x) = -\lambda' \int_a^b G(x, \xi) y^\circ(\xi) d\xi \quad (16)$$

where  $\lambda'$  is chosen such as to make  $y'(x)$  normalized, i.e.

$$\int_a^b [y'(x)]^2 dx = 1. \quad (17)$$

The initial function  $y^\circ$  is also normalized of course. Now in order to find out something about the nature of  $y'$ , let  $y^\circ(\xi)$  and  $G(x, \xi)$  in (16) be expanded by means of a generalized Fourier series as follows:

$$y^\circ(\xi) = a_1 \phi_1(\xi) + a_2 \phi_2(\xi) + \dots + a_k \phi_k(\xi) + \dots \quad (18)$$

$$G(x, \xi) = g_1(x) \phi_1(\xi) + g_2(x) \phi_2(\xi) + \dots + g_k(x) \phi_k(\xi) + \dots \quad (19)$$

where  $\phi_1, \phi_2, \dots, \phi_k$  are the eigenfunctions associated with the problem, and the coefficients  $a_k$  and  $g_k(x)$  are given by

$$a_k = \int_a^b y^\circ(\xi) \phi_k(\xi) d\xi \quad (20)$$

$$g_k(x) = \int_a^b G(x, \xi) \phi_k(\xi) d\xi. \quad (21)$$

Then substituting (18) and (19) into (16) and taking advantage of the orthonormal properties of the eigenfunctions  $\phi_k$ , one obtains for  $y'(x)$



$$y'(x) = -\lambda' \left[ a_1 g_1(x) + a_2 g_2(x) + \dots + a_k g_k(x) + \dots \right]. \quad (22)$$

But  $g_k(x)$  can be obtained in terms of  $\phi_k(x)$  by remembering that  $\phi_k$  is a solution of the differential equation

$$\begin{aligned} L \phi_k &= -\lambda_k \phi_k \\ V_i &= 0 \end{aligned} \quad (23)$$

where  $\lambda_k$  denotes the eigenvalue associated with  $\phi_k$ . Thus with the aid of the Green's function the solution  $\phi_k$  can be written as

$$\phi_k(x) = -\lambda_k \int_a^b G(x, \xi) \phi_k(\xi) d\xi. \quad (24)$$

Comparing (21) and (24) it can be seen that

$$g_k(x) = -\frac{\phi_k(x)}{\lambda_k}, \quad (25)$$

and thus  $y'$  can be written as

$$\begin{aligned} y'(x) &= -\lambda' \left[ a_1 \left( -\frac{\phi_1(x)}{\lambda_1} \right) + a_2 \left( -\frac{\phi_2(x)}{\lambda_2} \right) + \dots \right] \\ &= \frac{\lambda'}{\lambda_1} \left[ a_1 \phi_1(x) + a_2 \left( \frac{\lambda_1}{\lambda_2} \right) \phi_2(x) + \dots + a_k \left( \frac{\lambda_1}{\lambda_k} \right) \phi_k(x) + \dots \right]. \end{aligned} \quad (26)$$

Now if one compares the expanded expression for  $y'$  (26) with that for

$y^0$  (18), the effect of the iteration process can be seen. In going from  $y^0$  to  $y^1$  the values of the higher order "harmonics" have each been reduced by a factor of  $\frac{\lambda_i}{\lambda_k}$  with respect to the "fundamental." Here the terms harmonic and fundamental are used in the same generalized sense as the term Fourier series. As the eigenvalues have been arranged in ascending order, this ratio is always less than unity. The equations relating  $y^2$  to  $y^1$ , and  $y^3$  to  $y^2$  and so forth are similar to those relating  $y^1$  and  $y^0$ , so it can be said that the higher harmonics are reduced with respect to the fundamental by a factor of  $\frac{\lambda_i}{\lambda_k}$  with each step in the process. Clearly then, in the limit only the fundamental will remain. Or, if the fundamental is not present in  $y^0$ , the process will converge to the lowest order term which is present.

Next consider the limiting value of  $\lambda^n$ , the factor used to normalize each of the successive approximations. Assume for purposes of illustration that all of the terms of the expansion of  $y^0$  are zero up to the  $a_i$  term. That is, the  $i$ th term is the lowest one present. Then for large  $n$

$$y^n \approx y^{n-1} \approx \phi_i. \quad (27)$$

But  $y^n$  can also be written in terms of  $y^{n-1}$  by means of the recursion formula

$$\begin{aligned} y^n &= -\lambda^n \int_a^b G(x, \xi) y^{n-1}(\xi) d\xi \\ &\approx -\lambda^n \int_a^b G(x, \xi) \phi_i(\xi) d\xi. \end{aligned} \quad (28)$$

Now from (24)

$$\int_a^b G(x, \xi) \phi_i(\xi) d\xi = -\frac{1}{\lambda_i} \phi_i \quad (29)$$

Therefore, combining (27), (28) and (29) yields

$$\phi_i \approx (-\lambda^n) \left(-\frac{1}{\lambda_i} \phi_i\right)$$

or

$$\lambda^n \approx \lambda_i \quad (30)$$

Thus it is seen that the normalizing factor tends towards the eigenvalue corresponding to the particular eigenfunction obtained by iteration.

It is also of interest to note that  $\lambda^n$  approaches  $\lambda_i$  monotonically from above. For the sake of simplicity this will be demonstrated for the case of  $\lambda_1$ . A more general proof could follow along similar lines. Let  $y^{n-1}$  and  $y^n$  be written in expanded form as follows:

$$y^{n-1} = b_1 \phi_1 + b_2 \phi_2 + \dots \quad (31)$$

$$\begin{aligned} y^n &= -\lambda^n \int_a^b G(x, \xi) y^{n-1}(\xi) d\xi \\ &= \lambda^n \left[ \frac{b_1}{\lambda_1} \phi_1 + \frac{b_2}{\lambda_2} \phi_2 + \dots \right] \quad (32) \end{aligned}$$

Next, the normalized values of  $y^{n-1}$  and  $y^n$  may be written

$$\int_a^b [y^{n-1}]^2 dx = b_1^2 + b_2^2 + \dots \quad (33)$$

$$\int_a^b [y^n]^2 dx = [\lambda^n]^2 \left[ \left(\frac{b_1}{\lambda_1}\right)^2 + \left(\frac{b_2}{\lambda_2}\right)^2 + \dots \right], \quad (34)$$

and then recalling that the normalized values of both  $y^{n-1}$  and  $y^n$  must be unity, the above expressions may be set equal with the result

$$\left[ \frac{\lambda^n}{\lambda_1} \right]^2 = \frac{b_1^2 + b_2^2 + b_3^2 + \dots}{b_1^2 + \left(\frac{\lambda_1}{\lambda_2} b_2\right)^2 + \left(\frac{\lambda_1}{\lambda_3} b_3\right)^2 + \dots} \quad (35)$$

This ratio is always greater than unity because  $\lambda_1 < \lambda_2 < \lambda_3 \dots$ . Thus the approximate value for the eigenvalue is always larger than the correct value. Also, as the higher harmonics ( $b_2, b_3, \dots$ ) decrease as  $n$  becomes larger, it can be seen that this ratio approaches unity getting closer with each iterative step.

It has thus been shown that any of the eigenfunctions and corresponding eigenvalues for the general Sturm-Liouville problem can be obtained by this method. One starts by assuming an initial  $y^0$  satisfying the boundary conditions and then uses this to obtain a  $y^1$ , and the  $y^1$  to obtain a  $y^2$ , and so forth until the desired accuracy is obtained. Normally, if  $y^0$  has been chosen in an arbitrary manner, one would expect the fundamental and all harmonic terms to be present in the expansion. This being the case the process would converge to the fundamental eigenfunction  $\phi_1$ . Once this has been determined the fundamental component of  $y^0$  can be subtracted out of  $y^0$  and a new initial function obtained which contains no fundamental. This can then be used to find  $\phi_2$ , and

the process repeated to find  $\phi_3$ , and so forth. The magnitude of the fundamental term present in  $y^0$  can be obtained from the usual expression for the coefficients of a Fourier series, i.e.

$$a_1 = \int_a^b y^0(x) \phi_1(x) dx \quad (36)$$

A similar expression may be written for the other components.

Before going on to an example something might be said about the rate at which this process converges. Nothing of a general nature has been developed on this point. However, some feel for the rate of convergence may be had by observing the factor by which each of the higher order terms is reduced relative to the fundamental with each step of the process. For example, when obtaining the fundamental the  $k$ th order harmonic term is reduced by a factor of  $\frac{\lambda_1}{\lambda_k}$  with each step. Thus if the eigenvalues of the problem are widely separated, one would expect rather rapid convergence. On the other hand if two of the eigenvalues are close together, say  $\lambda_1$  and  $\lambda_2$  for example, then one would expect slow convergence as the second term would only be modified by the ratio  $\frac{\lambda_2}{\lambda_1}$  with each iterative step.

#### D. Harmonic Equation Example

The example chosen is rather trivial in that an analytical solution is readily available. However, it should serve to illustrate the method and is analogous to the two-dimensional wave equation problem encountered

later. The differential system to be considered is

$$\begin{aligned} \frac{d^2 y}{dx^2} + \lambda y &= 0, & 0 < x < \pi \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned} \quad (37)$$

The normalized set of eigenfunctions for this problem will be recognized as  $\sqrt{\frac{2}{\pi}} \sin x$ ,  $\sqrt{\frac{2}{\pi}} \sin 2x$ ,  $\sqrt{\frac{2}{\pi}} \sin 3x$ , ... and the corresponding eigenvalues 1, 4, 9, ... .

The differential equation of (37) can be written in the form

$$L y = -\lambda y \quad (38)$$

where  $L$  denotes the simple operator  $\frac{d^2}{dx^2}$ . The first step will be to obtain the Green's function for the corresponding homogeneous problem

$$\begin{aligned} L y &= 0 \\ y(0) &= 0 \\ y(\pi) &= 0. \end{aligned} \quad (39)$$

This is well known<sup>a</sup> and is given by

$$\begin{aligned} G(x, \xi) &= -\frac{x}{\pi} (\pi - \xi), & x \leq \xi \\ &= -\left(\frac{\pi - x}{\pi}\right) \xi, & x \geq \xi. \end{aligned} \quad (40)$$

---

<sup>a</sup>A general expression for the Green's function for a second order system is given in Ince (3) p. 257.

Next a  $y^0$  function must be assumed which will satisfy the boundary conditions. As a matter of convenience a rectangular function as shown in Fig. 1 will be used for  $y^0$ . This function is discontinuous at 0 and  $\pi$ , but this will not present a serious problem as all operations on  $y^0$  involve integration rather than differentiation.

Now  $y^1$  is obtained from the equation

$$y^1(x) = -\lambda^1 \int_0^\pi G(x, \xi) y^0(\xi) d\xi \quad (41)$$

where  $\lambda^1$  is chosen such as to normalize  $y^1$ . Normally, in a more complex problem one would have to perform the integration numerically. However, in this case  $y^0$  was purposely chosen such that (41) could be integrated readily. A similar equation applies for obtaining  $y^2$  from  $y^1$ . The initial function  $y^0$  and the first two successive approximations are sketched in Fig. 1. The function  $y^2$  is so close to the exact solution

$\sqrt{\frac{2}{\pi}} \sin x$  as to make them indistinguishable in the sketch. Also,

these functions and the values of  $\lambda^1$  and  $\lambda^2$  are tabulated in Table 1.

It can be seen that in this case a relatively good approximation is obtained in just two steps, even though the initial function  $y^0$  is a poor approximation. Notice that the error in the eigenvalue is less than 1 per cent. This rather rapid convergence is due to the relatively wide separation of  $\lambda_1$  from the other eigenvalues of the problem.

Another somewhat similar example involving the harmonic equation is the differential system

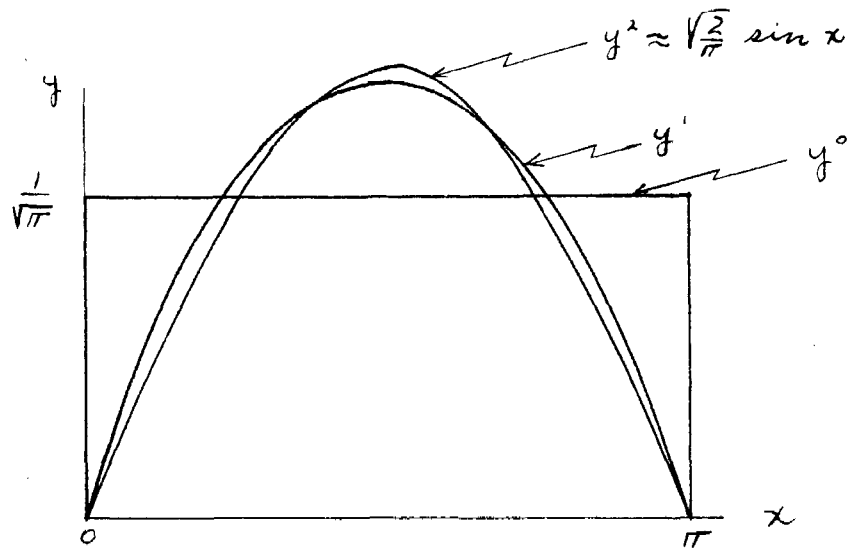


Fig. 1 Sketch of  $y^0$ ,  $y^1$  and  $y^2$ .

Table 1. Equations for Successive Approximations

$n$	$y^n$	$\lambda^n$
0	$\frac{1}{\sqrt{\pi}}$	—
1	$3.09 \frac{x}{\pi} (1 - \frac{x}{\pi})$	1.11
2	$2.55 \frac{x}{\pi} [1 - 2(\frac{x}{\pi})^2 + (\frac{x}{\pi})^3]$	1.006
	. . . . .	
$\infty$	$\frac{2}{\sqrt{\pi}} \sin x$	1.000



$$\frac{d^2 y}{dx^2} + \lambda y = 0 \quad (42)$$

$$y'(0) = 0$$

$$y'(\pi) = 0$$

where the prime denotes the derivative. This problem is analogous to a wave equation problem encountered later in which the normal derivative rather than the function itself is set equal to zero on the boundary. It might appear at first glance that the procedure for solving this problem would be the same as that used in the previous example. It is not, though, because the Green's function for the corresponding homogeneous problem in this case,

$$\frac{d^2 y}{dx^2} = 0 \quad (43)$$

$$y'(0) = 0$$

$$y'(\pi) = 0,$$

does not exist. This is due to the fact that (43) has an eigenvalue at the origin. This is to say that the homogeneous system of (43) is compatible and admits a nontrivial solution, namely a constant, which satisfies the boundary conditions and the continuity requirements at every point within the interval. One would expect to encounter difficulty in such a case.

One way of obviating the difficulty is to add and subtract  $\delta y$  to the differential equation and then regroup terms as follows:

$$\left( \frac{d^2 y}{dx^2} + \delta y \right) + \Lambda y = 0 \quad (44)$$

where

$$\Lambda = \lambda - \delta. \quad (45)$$

Now a Green's function will exist for the system

$$\begin{aligned} \frac{d^2 y}{dx^2} + \delta y &= 0 \\ y'(0) &= 0 \\ y'(\pi) &= 0, \end{aligned} \quad (46)$$

and one can proceed as before with  $\Lambda$  as the parameter of the problem.

Once the eigenvalues  $\Lambda_1, \Lambda_2, \dots$  have been determined, then the corresponding ones in the original problem can be obtained from (45).

The constant  $\delta$  may be chosen to be any convenient value except the eigenvalues of the original problem. These values must be avoided or the system of (46) would become compatible and the original difficulty would arise again. However, there is nothing to prevent choosing  $\delta$  to be infinitesimally small, and in this case some simplification results. Briefly, without going into the details of the solution, the Green's function for small  $\delta$  approaches

$$\begin{aligned} G(x, \xi) &\approx \frac{1}{\delta \pi} - \frac{1}{2\pi} \left[ (\xi - \pi)^2 + x^2 \right], \quad x \leq \xi \\ &\approx \frac{1}{\delta \pi} - \frac{1}{2\pi} \left[ (x - \pi)^2 + \xi^2 \right], \quad x \geq \xi. \end{aligned} \quad (47)$$

Note that  $G(x, \xi)$  consists of a constant term plus a variable one, and as  $\delta$  approaches zero the constant term approaches infinity. This, however, presents no difficulty if the initial  $y^0$  function is chosen such that it has no constant component. Then the only significant part of the Green's function is the variable part which is not dependent upon  $\delta$ .

The resulting eigenfunctions of the problem are  $\sqrt{\frac{2}{\pi}} \cos x$ ,  $\sqrt{\frac{2}{\pi}} \cos 2x$ ,  $\sqrt{\frac{2}{\pi}} \cos 3x$ , ... .

Thus, even though the Green's function for the original problem does not exist, the problem can still be solved by adding and subtracting a term in the differential equation such that a new problem is formed which does not have an eigenvalue at zero.

### III. THE WAVE GUIDE

#### A. Mathematical Formulation of the Problem

At this point a resume of classical wave guide theory will be presented. Of necessity, the presentation must be brief, and the reader is referred to one of the many texts on the subject for a more detailed treatment. The approach used here will be essentially the same as that given in Slater (12).

The assumption will be made from the beginning that the walls of the guide are perfectly conducting, and that its interior is filled with a lossless homogeneous dielectric material. The cross-sectional shape of the bounding surface of the guide may be arbitrary as shown in Fig. 2, and the coordinate system will be chosen with the  $z$  axis in the longitudinal direction.

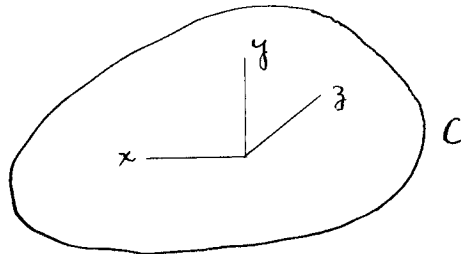


Fig. 2 Coordinate System for Wave Guide

The field within the guide must satisfy Maxwell's equations, the divergence relationships and the boundary conditions listed.

$$\begin{aligned}\nabla \times \bar{\mathcal{E}} &= -\mu \frac{\partial \bar{\mathcal{H}}}{\partial t} \\ \nabla \times \bar{\mathcal{H}} &= \epsilon \frac{\partial \bar{\mathcal{E}}}{\partial t}\end{aligned}\tag{48}$$

$$\begin{aligned}\nabla \cdot \bar{\mathcal{E}} &= 0 \\ \nabla \cdot \bar{\mathcal{H}} &= 0.\end{aligned}\tag{49}$$

Boundary conditions:

$$\left. \begin{aligned}\bar{n} \times \bar{\mathcal{E}} &= 0 \\ \bar{n} \times \bar{\mathcal{H}} &= 0\end{aligned} \right\} \text{on } C\tag{50}$$

where  $\bar{n}$  denotes the unit normal vector along  $C$ , and  $\bar{\mathcal{E}}$  and  $\bar{\mathcal{H}}$  denote functions of  $x$ ,  $y$ ,  $z$ , and  $t$  at this point. If time dependence of the form  $e^{j\omega t}$  is assumed, Maxwell's equations become

$$\begin{aligned}\nabla \times \bar{\mathcal{E}} &= -j\omega\mu\bar{\mathcal{H}} \\ \nabla \times \bar{\mathcal{H}} &= j\omega\epsilon\bar{\mathcal{E}}\end{aligned}\tag{51}$$

where

$$\begin{aligned}\bar{\mathcal{E}} &= \bar{E}(x, y, z) e^{j\omega t} \\ \bar{\mathcal{H}} &= \bar{H}(x, y, z) e^{j\omega t}.\end{aligned}\tag{52}$$

Taking the curl of both sides of (51) and combining the two yields

$$\begin{aligned}\nabla \times \nabla \times \bar{E} - \omega^2 \mu \epsilon \bar{E} &= 0 \\ \nabla \times \nabla \times \bar{H} - \omega^2 \mu \epsilon \bar{H} &= 0 ,\end{aligned}\tag{53}$$

and if  $\nabla \cdot \bar{E}$  and  $\nabla \cdot \bar{H}$  are both zero the above equations become

$$\begin{aligned}\nabla^2 \bar{E} + \omega^2 \mu \epsilon \bar{E} &= 0 \\ \nabla^2 \bar{H} + \omega^2 \mu \epsilon \bar{H} &= 0 .\end{aligned}\tag{54}$$

These are the vector wave equations.

Further, if the wave is assumed to propagate in the  $z$  direction, the  $z$  dependence is of the form  $e^{-\gamma z}$ , and the wave equations then become

$$\begin{aligned}\nabla_t^2 \bar{E}' + k_c^2 \bar{E}' &= 0 \\ \nabla_t^2 \bar{H}' + k_c^2 \bar{H}' &= 0\end{aligned}\tag{55}$$

where

$$\begin{aligned}\bar{E} &= \bar{E}'(x, y) e^{-\gamma z} \\ \bar{H} &= \bar{H}'(x, y) e^{-\gamma z}\end{aligned}\tag{56}$$

$$\nabla_t^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)\tag{57}$$

$$k_c^2 = \gamma^2 + \omega^2 \mu \epsilon .\tag{58}$$

At this point it will be convenient to resolve  $\bar{E}'$  and  $\bar{H}'$  in terms of

their transverse and longitudinal components. Let

$$\begin{aligned}\bar{E}' &= \bar{E}_t + \bar{k} E_z \\ \bar{H}' &= \bar{H}_t + \bar{k} H_z\end{aligned}\tag{59}$$

where  $\bar{k}$  denotes the unit vector in the  $\bar{j}$  direction and  $\bar{E}_t$  and  $E_z$  denote the transverse and longitudinal components of  $\bar{E}'$  respectively.

The primes have been dropped at this point, but this should cause no confusion if one remembers that the resolution did not take place until after the  $\bar{j}$  dependence had been removed.

Now, if the equations of (59) are substituted in the wave equations of (55) the following equations result.

$$\begin{aligned}(\nabla_t^2 \bar{E}_t + k_c^2 \bar{E}_t) + (\nabla_t^2 E_z + k_c^2 E_z) \bar{k} &= 0 \\ (\nabla_t^2 \bar{H}_t + k_c^2 \bar{H}_t) + (\nabla_t^2 H_z + k_c^2 H_z) \bar{k} &= 0.\end{aligned}\tag{60}$$

In each equation the terms in parentheses are at right angles to each other, and therefore both terms must be zero. Thus  $E_z$  and  $H_z$  must satisfy the scalar wave equations

$$\begin{aligned}\nabla_t^2 E_z + k_c^2 E_z &= 0 \\ \nabla_t^2 H_z + k_c^2 H_z &= 0.\end{aligned}\tag{61}$$

Also, by substituting the equations of (59) into Maxwell's equations and, after considerable algebraic manipulation, one can obtain the trans-

verse components of the field in terms of the longitudinal components as follows<sup>a</sup>.

$$\begin{aligned}\bar{E}_t &= -\frac{\gamma}{k_c^2} \nabla_t E_z + \frac{j\beta_0}{k_c^2} \left[ \bar{k} \times \nabla_t (\eta H_z) \right] \\ \bar{H}_t &= -\frac{\gamma}{k_c^2} \nabla_t H_z - \frac{j\beta_0}{k_c^2} \left[ \bar{k} \times \nabla_t \left( \frac{E_z}{\eta} \right) \right],\end{aligned}\quad (62)$$

where  $\beta_0$  and  $\eta$  are the "free space" phase shift constant and wave impedance respectively, i.e.

$$\beta_0 = \omega \sqrt{\mu\epsilon} \quad (63)$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}. \quad (64)$$

Equations (62) are particularly significant. From them it can be seen that once  $E_z$  and  $H_z$  have been determined, the field within the guide is completely specified. Usually two separate cases are considered, one where  $E_z = 0$  and the other where  $H_z = 0$ . Then any general propagating field can be represented as a superposition of the two cases. The case of zero  $E_z$  is called the transverse electric or TE mode and the zero  $H_z$  case the transverse magnetic or TM mode.

The problem has now been reduced to one of finding  $E_z$  and  $H_z$ , both of which must satisfy the scalar wave equation. The boundary condition for

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<sup>a</sup>See Ramo and Whinnery (10) pp. 344-354 or Slater (12) pp. 6-8.



$E_z$  is obvious, as  $E_z$  is tangent to the surface of a perfect conductor at the boundary and thus must be zero along  $C$ . The corresponding relationship for  $H_z$  is not so obvious because  $H_z$ , being parallel to the bounding surface, satisfies the original  $\bar{n} \cdot \bar{H} = 0$  requirement. The answer lies in equations (62) for the transverse components which must also satisfy the  $\bar{n} \times \bar{E} = 0$  and  $\bar{n} \cdot \bar{H} = 0$  requirements. It can be seen from (62) that if  $H_z$  has a normal derivative at the boundary,  $\bar{E}_t$  will have a tangential component there which would violate the  $\bar{n} \times \bar{E} = 0$  requirement. Thus the normal derivative of  $H_z$  must be zero along  $C$ .

As yet, nothing has been said about the zero divergence criteria. The mere fact that the fields satisfy the wave equation does not insure zero divergence for  $\bar{E}$  and  $\bar{H}$ . However, in this case the equations for the transverse components of the field were derived from Maxwell's equations. This is sufficient to make  $\text{div } \bar{E}$  and  $\text{div } \bar{H}$  zero, as each is the curl of another vector and the divergence of the curl of a vector is identically zero.

In summary, the wave guide problem reduces to two separate boundary value problems:

1. TE case

$$\begin{aligned} \nabla_t^2 H_z + k_c^2 H_z &= 0 \\ \frac{\partial H_z}{\partial n} &= 0 \quad \text{on } C \end{aligned} \tag{65}$$

2. TM case

$$\begin{aligned} \nabla_t^2 E_z + k_c^2 E_z &= 0 \\ E_z &= 0 \quad \text{on } C. \end{aligned} \tag{66}$$

Solutions to the above equations exist only for discrete values of  $k_c$ , which in general are not the same for both cases, of course. These values of  $k_c$  are quite important in wave guide work as they, along with the frequency, determine the propagating characteristics of the system. In the following section an iterative method for finding these allowable values of  $k_c$  and the associated solutions will be presented.

## B. Solution by Successive Approximations

### 1. The general case

It will now be shown that the wave equations of the previous section can be solved, at least in principle, by the method of successive approximations. The procedure to be used here will be similar to that used in section II, and while discussing the generalities of the method both cases will be considered simultaneously. The differential system to be solved is of the form

$$\begin{aligned} \nabla^2 \phi + [\lambda - q(x,y)] \phi &= 0 \\ \text{or } \left. \begin{array}{l} \phi = 0 \\ \frac{\partial \phi}{\partial n} = 0 \end{array} \right\} \text{ on } C \end{aligned} \quad (67)$$

where  $\nabla^2$  is now understood to be the two dimensional Laplacian operator. This problem with  $q=0$  is discussed in some detail by Slater (12), and it is shown there that the solutions form an orthogonal set which can be used for the expansion of any arbitrary function which satisfies the boundary

condition\*. The addition of the  $q(x, y)$  term will not invalidate any of this theory. This can be seen by recalling the procedure used for showing that the solutions are orthogonal. If  $\phi_n$  and  $\phi_m$  are any two nondegenerate solutions corresponding to  $\lambda_n$  and  $\lambda_m$  then

$$\begin{aligned}\nabla^2 \phi_n + [\lambda_n - q] \phi_n &= 0 \\ \nabla^2 \phi_m + [\lambda_m - q] \phi_m &= 0.\end{aligned}$$

Multiplying the first by  $\phi_m$  and the second by  $\phi_n$  and subtracting yields

$$(\phi_m \nabla^2 \phi_n - \phi_n \nabla^2 \phi_m) + (\lambda_n - \lambda_m) \phi_n \phi_m = 0. \quad (68)$$

Notice that the terms involving  $q$  cancel. Now, if the equation is integrated over the cross-sectional area, and the two-dimensional form of Green's formula is used to transform the first term to a line integral around  $C$ , it can be seen from this and the boundary conditions that  $\phi_m$  and  $\phi_n$  are orthogonal. Inclusion of the  $q$  term in (67) is necessary, as the same trouble will be encountered here as before with regard to the existence of a Green's function in the case of the  $\frac{\partial \phi}{\partial n} = 0$  boundary condition.

Returning to the original problem, it will be assumed that  $q$  is such that there is no eigenvalue at the origin and that the solutions are

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\*The restriction that the arbitrary function must satisfy the boundary condition is not a necessary one. However, by making this restriction, the question of continuity at the boundary is avoided.

nondegenerate, i.e. a different eigenvalue will be associated with each eigenfunction. First let equation (67) be written in the form

$$[\nabla^2 - q] \phi = -\lambda \phi \quad (69)$$

and let  $G(x, y, \xi, \eta)$  be the Green's function for the system

$$\left. \begin{aligned} [\nabla^2 - q] \phi &= 0 \\ \phi &= 0 \\ \text{or } \frac{\partial \phi}{\partial n} &= 0 \end{aligned} \right\} \text{ on } C. \quad (70)$$

As before, the successive approximations  $\phi^1, \phi^2, \dots$  will be formed from the differential equations

$$\begin{aligned} [\nabla^2 - q] \phi^1 &= -\lambda^1 \phi^0 \\ [\nabla^2 - q] \phi^2 &= -\lambda^2 \phi^1 \\ &\dots \\ [\nabla^2 - q] \phi^n &= -\lambda^n \phi^{n-1}, \end{aligned} \quad (71)$$

where  $\phi^0$  is the initial function chosen such as to satisfy the boundary conditions.

Now the solution for  $\phi^1$  can be written in terms of the Green's function as

$$\phi^1(x, y) = -\lambda^1 \iint_S G(x, y, \xi, \eta) \phi^0(\xi, \eta) d\xi d\eta \quad (72)$$

where the integration is over the cross-sectional area  $S$ . Both  $\phi^0$

and  $G$  can be expanded in terms of the eigenfunctions of (67) as follows:

$$\begin{aligned} \phi^{\circ}(\xi, \eta) = a_1 \psi_1(\xi, \eta) + a_2 \psi_2(\xi, \eta) + \dots \\ + a_n \psi_n(\xi, \eta) + \dots \end{aligned} \quad (73)$$

$$\begin{aligned} G(x, y, \xi, \eta) = g_1(x, y) \psi_1(\xi, \eta) + g_2(x, y) \psi_2(\xi, \eta) + \dots \\ + g_n(x, y) \psi_n(\xi, \eta) + \dots \end{aligned} \quad (74)$$

where  $\psi_1, \psi_2, \dots$  are the normalized eigenfunctions of (67) and  $a_n$  and  $g_n$  are given by

$$a_n = \iint_S \phi^{\circ}(\xi, \eta) \psi_n(\xi, \eta) d\xi d\eta \quad (75)$$

$$g_n(x, y) = \iint_S G(x, y, \xi, \eta) \psi_n(\xi, \eta) d\xi d\eta. \quad (76)$$

Then, substituting the expanded forms for  $\phi^{\circ}$  and  $G$  into (72) and taking advantage of the orthonormal properties of  $\psi_1, \psi_2, \dots$  leads to

$$\phi'(x, y) = -\lambda' [a_1 g_1(x, y) + a_2 g_2(x, y) + \dots + a_n g_n(x, y) + \dots]. \quad (77)$$

However, the eigenfunction  $\psi_n$  can be written in integral equation form as

$$\psi_n(x, y) = -\lambda_n \iint_S G(x, y, \xi, \eta) \psi_n(\xi, \eta) d\xi d\eta. \quad (78)$$

It can be seen by comparing (76) and (78) that

$$g_n(x, y) = - \frac{\psi_n(x, y)}{\lambda_n}, \quad (79)$$

and substituting this into (77) gives

$$\phi' = \lambda' \left[ \frac{a_1}{\lambda_1} \psi_1 + \frac{a_2}{\lambda_2} \psi_2 + \cdots + \frac{a_n}{\lambda_n} \psi_n + \cdots \right]. \quad (80)$$

The arguments with regard to convergence are the same here as in the one-dimensional case previously analyzed. As a matter of fact the whole derivation is essentially the same, except that the one-dimensional functions of the Sturm-Liouville problem have been replaced with two-dimensional ones. The whole matter of extending the theory to the two-dimensional case hinges upon the validity of expanding a function in terms of the eigenfunctions of the problem. No formal justification of this will be given here. However, the problem has been investigated rather thoroughly by Titchmarsh (14), and he states that such an expansion is valid, at least for the  $\phi = 0$  boundary condition case.

## 2. Green's functions for the TM and TE cases

It has been shown in principle at least, that the wave equation can be solved by the method of successive approximations. However, before the method can be applied the Green's function for the problem at hand must be determined, and this is usually not an easy matter. In a sense, a difficult problem, that of solving the wave equation, has merely been traded

for an almost equally difficult one, that of finding the Green's function. As the TM case is the more straightforward of the two, it will be considered first.

In the TM case  $E_z$  must be zero on the boundary. Thus the Green's function for a differential system of the form

$$\begin{aligned}\nabla^2 \phi &= 0 \\ \phi &= 0 \quad \text{on } C\end{aligned}\tag{81}$$

must be found. This Green's function is relatively well known and is discussed in Phillips (8) and other texts dealing with potential theory. However, the approach used here will differ somewhat from that of Phillips and will be essentially the same as that used before in section II. The solution of the equation

$$\nabla^2 \phi' = -\lambda' \phi^0\tag{82}$$

has been written in the form

$$\phi'(x, y) = \iint_S G(x, y, \xi, \eta) [-\lambda' \phi^0(\xi, \eta)] d\xi d\eta.\tag{83}$$

This will be considered the defining equation for  $G(x, y, \xi, \eta)$ , i.e.

$G$  must be such that (83) will be true. If one operates on both sides of (83) with  $\nabla^2$  it can be seen that  $\nabla^2 G$  must have the properties of a delta function just as in the one-dimensional case. Thus  $G$  must satisfy the homogeneous equation (81) and the boundary conditions except at the point  $(\xi, \eta)$ . At this point it must be such that the integral of  $\nabla^2 G$

over a small area including the point will be unity. Recalling a little electrostatic theory, it can be seen that the potential due to a fine line of charge at  $(\xi, \eta)$  (and the associated induced charge on the conducting boundary, of course) is just the type of function needed. This function satisfies Laplace's equation except at  $(\xi, \eta)$ , is zero on the boundary and is such that the integral of  $\nabla^2 G$  over a small region including  $(\xi, \eta)$  is a constant. It only remains to find the linear charge density required to make this integral unity.

Consider a cylindrical surface of infinitesimal radius  $\delta$ , of unit length and coaxial with respect to the fine line of charge at  $(\xi, \eta)$ . The potential in this region will be approximately that due to the line of charge and is given by

$$G \approx -\frac{Q}{2\pi\epsilon} \ln r + \text{constant} \quad (84)$$

where  $r$  is the distance from the point  $(\xi, \eta)$  to  $(x, y)$  and  $Q$  is the charge per unit length in the MKS system of units. Then  $\nabla^2 G$  is given by

$$\nabla^2 G = -\frac{Q}{2\pi\epsilon} \nabla^2 \ln r = -\frac{Q}{2\pi\epsilon} \nabla \cdot \frac{1}{r} \bar{I}_r. \quad (85)$$

The surface integral of this over one end of the cylinder, which is of interest here, is the same as the volume integral because of the unit length of the cylinder. This volume integral can then be transformed to a surface integral using the divergence theorem and the result is



$$\iint_{\pi \delta^2} \nabla^2 G \, ds = \iiint_{\pi \delta^2(i)} -\frac{Q}{2\pi\epsilon} \nabla \cdot \frac{1}{r} \bar{1}_r \, dv = -\frac{Q}{\epsilon} . \quad (86)$$

Thus  $Q$  must equal  $-\epsilon$  and  $G$  becomes  $\frac{1}{2\pi} \ln r$  in the neighborhood of  $(\xi, \eta)$ .

The Green's function for the TE case ( $\frac{\partial \phi}{\partial n} = 0$  on  $C$ ) is not quite so obvious as that of the TM case just described. The same problem is encountered here as in the example of section II. The Green's function for the system

$$\begin{aligned} \nabla^2 \phi &= 0 \\ \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } C \end{aligned} \quad (87)$$

does not exist because the system is compatible, i.e.  $\phi = \text{constant}$  is a perfectly satisfactory nontrivial solution, at least in the mathematical sense of the word. Thus the problem must be modified. Let  $\delta\phi$  be added and subtracted to the original equation and terms regrouped as follows:

$$\begin{aligned} \nabla^2 \phi + \delta\phi &= -\Lambda \phi \\ \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } C \end{aligned} \quad (88)$$

where

$$\Lambda = \lambda - \delta .$$

Now a Green's function will exist for the homogeneous problem

$$\begin{aligned}\nabla^2 \phi + \delta \phi &= 0 \\ \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } C\end{aligned}\tag{89}$$

providing  $\delta$  is not an eigenvalue of the problem.

Here, as before, some simplification of the problem results from choosing  $\delta$  infinitesimally small. This being the case, the Green's function must satisfy the homogeneous differential equation (89) within the region  $S$  except at the point  $(\xi, \eta)$ . Recalling the similar problem of section II, one would expect the  $\delta G$  term of (89) (with  $G$  substituted for  $\phi$ ) to approach a constant as  $\delta$  goes to zero. This will be assumed at this point, and it will be shown that a function compatible with this assumption and all of the other requirements of a Green's function can be found. Although not absolutely necessary the function will be described in terms of electrostatics in order to provide a physical picture for better understanding.

The Green's function will be fabricated in four steps beginning at the point  $(\xi, \eta)$ . At this point it must have the usual delta function properties, and thus this calls for a fine line of charge of  $-\epsilon$  coulombs per meter at point  $(\xi, \eta)$  as in the previous case. Then, if the  $\delta G$  term in (89) is to be a constant, a uniform charge density must exist within the region  $S$  as (89) is just Poisson's equation. Also, as the normal derivative of  $G$  is to be zero on the boundary, there can be no electrostatic flux impinging on  $C$ . Thus by Gauss' law the total distributed charge must be equal and opposite to that of the fine line at

$(\xi, \eta)$ . This, along with knowing that the charge is distributed uniformly, enables one to find the charge density.

So far, the potential function will satisfy the differential equation (89) and will have the appropriate delta function properties at  $(\xi, \eta)$ , but it will not satisfy the boundary conditions. Therefore, a function satisfying Laplace's equation within  $C$  and having a normal derivative on  $C$  opposite to that of the potential due to the fine line and distributed charges must be found and added in order to satisfy the boundary conditions. This constitutes the Neumann problem or the second boundary value problem of potential theory, and such a function can be found within a constant<sup>a</sup>. The physical distribution of charges giving rise to such a function is a double layer of charges on the boundary<sup>b</sup>. Finally, a constant term must be added which approaches infinity as  $\delta$  goes to zero. This is necessary to make the  $\delta G$  term equal to the distributed charge density constant.

Thus the final potential function is a superposition of the potential due to:

1. A fine line of negative charge at  $(\xi, \eta)$
2. Uniformly distributed positive charge within  $C$
3. Double layer of charge on  $C$
4. A constant term approaching infinity as  $\delta$  goes to zero.

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<sup>a</sup>A detailed explanation of this problem can be found in Phillips (8) p. 170.

<sup>b</sup>A discussion of this can also be found in Phillips (8) p. 140.

It can be seen that if the initial  $\phi^0$  has no constant term, then the constant term of the Green's function is of no importance and can be omitted. This is an important point; choosing  $\delta$  infinitesimally small certainly does not simplify matters unless  $\phi^0$  is chosen such as to have no constant term.

### 3. Comments on the practical aspects of the method and degeneracy

In studying the theory just discussed one is immediately struck with the immensity of the task of finding the Green's function for the general problem. If one were to do this by means of some numerical process, the first step would probably be to subdivide the region within  $C$  into many smaller ones and proceed on an incremental basis. The electrostatic problems arising from placing a fine line of charge in each of the incremental regions must be solved, and this gives rise to as many separate electrostatic problems as there are incremental regions. Each of these problems would, in turn, have to be solved by some sort of numerical process. And further, all of this must be done before the iterative process can even be begun. At this point all of this may seem a little hopeless as a practical means of obtaining a solution, at least without the aid of a computer. However, there is a possibility that some short cuts can be used by approaching the problem from a little different viewpoint, and this will be discussed in a later section.

Also, the possibility of degeneracy in solutions has been carefully avoided up to this point. Degeneracy occurs when two or more independent

solutions have the same eigenvalues, and this usually happens when there is some sort of symmetry to the bounding curve  $C$ . An example of this would be the case where  $C$  is a square. Here the  $TM_{mn}$  and  $TM_{nm}$  modes have identical eigenvalues. This problem of degenerate solutions is discussed with regard to vibrating membrane problems in Weinstock (15), and, as the same equations apply in both the wave guide and membrane problems, the theory applies here. Weinstock shows that the number of solutions having the same eigenvalue must be finite, and further that linear combinations of these can be formed in such a way that these combinations are orthogonal with respect to each other (and the other eigenfunctions, of course).

The question now arises as to how degeneracy might affect the method of successive approximations just described. Once it has been established that linear combinations of the degenerate modes can be formed which are orthogonal, then it can be safely concluded that the eigenfunction expansions used in proving convergence are still valid in the degenerate case. Nothing then will be affected except the final conclusions. It will be recalled that the method was deduced to converge to the lowest order mode, say  $\psi_k$ , because each of the higher order ones, say  $\psi_n$ , was reduced by a factor of  $\frac{\lambda_k}{\lambda_n}$  relative to  $\psi_k$  with each step of the iterative process. Now if this lowest order mode is a doublet for example, then both of these terms would remain in the same ratio with respect to each other, while the others go to zero in the limit. Thus, the method would converge to a normalized linear combination of the two with their

ratio being the same as in the original assumed  $\psi^o$  function. The eigenvalue obtained would be correct but the two resulting modes are inseparable using this method.

This is not surprising, as one might expect something like this to occur regardless of the method used, since the way in which linear combinations are formed does not lead to a unique set of orthogonal combinations. Certainly the same thing holds true for the Ritz method where the technique depends on the minimal properties of the eigenfunctions. That is, any normalized linear combination of degenerate modes has the same minimal property as any other linear combination.

One further comment might be made at this point with regard to extending the method to the three-dimensional differential equation

$$\nabla^2 \phi + [\lambda - q(x, y, z)] \phi = 0. \quad (90)$$

This equation is of considerable importance in mathematical physics. As well as being of interest in classical physics, it will be recognized as the famous Schrodinger equation of quantum mechanics. All of the theory just discussed will apply equally well to this equation after making a few modifications to account for three independent variables rather than two. The modifications are rather obvious and will not be pursued further.

## IV. THE RESONANT CAVITY

## A. General Remarks

The same assumptions with regard to perfectly conducting walls and homogeneous lossless dielectric material will be made here as in the wave guide problem. It has been shown previously that if time dependence of the form  $e^{j\omega t}$  is assumed, Maxwell's equations take on the form

$$\begin{aligned}\nabla \times \bar{E} &= -j\omega\mu\bar{H} \\ \nabla \times \bar{H} &= j\omega\epsilon\bar{E}\end{aligned}\tag{91}$$

where both  $\bar{E}$  and  $\bar{H}$  are functions of  $x$ ,  $y$  and  $z$ . In addition, the field must satisfy the zero divergence relationships and the boundary conditions, i.e.

$$\begin{aligned}\nabla \cdot \bar{E} &= 0 \\ \nabla \cdot \bar{H} &= 0\end{aligned}\tag{92}$$

$$\left. \begin{aligned}\bar{n} \times \bar{E} &= 0 \\ \bar{n} \cdot \bar{H} &= 0\end{aligned} \right\} \text{ on } S\tag{93}$$

where  $S$  is the closed boundary surface. These equations define the problem, and if an  $\bar{E}$  and  $\bar{H}$  can be found which will satisfy these equations, that is all that is required. It might be pointed out that for the purposes of this problem, the zero divergence equations are superfluous

in that they follow directly from Maxwell's equations, i.e. both  $\bar{E}$  and  $\bar{H}$  are the curl of a vector and thus their divergence is zero.

If the curl of both sides of (91) is taken and the equations combined, the result is

$$\begin{aligned}\nabla \times \nabla \times \bar{E} - k^2 \bar{E} &= 0 \\ \nabla \times \nabla \times \bar{H} - k^2 \bar{H} &= 0\end{aligned}\tag{94}$$

where

$$k^2 = \omega^2 \mu \epsilon .\tag{95}$$

Further, if  $\bar{E}$  and  $\bar{H}$  have zero divergence the expressions of (94) can be written as

$$\begin{aligned}\nabla^2 \bar{E} + k^2 \bar{E} &= 0 \\ \nabla^2 \bar{H} + k^2 \bar{H} &= 0 .\end{aligned}\tag{96}$$

These are the vector wave equations. Now the question arises, which set of equations should be solved, (94) or (96), or does it make any difference? An attempt will be made to answer this question.

Consider (94) first. If the divergence of both sides is taken, it can be seen that  $\nabla \cdot \bar{E}$  and  $\nabla \cdot \bar{H}$  are both zero, and therefore any  $\bar{E}$  and  $\bar{H}$  satisfying (94) must also satisfy the zero divergence conditions and thus Maxwell's equations. Also, only one of the equations must be solved, say the  $\bar{E}$  equation, as the expression for the other will follow directly from Maxwell's equations. Therefore, the problem reduces to solving differential equation



$$\begin{aligned}\nabla \times \nabla \times \bar{E} - k^2 \bar{E} &= 0 \\ \bar{n} \times \bar{E} &= 0 \quad \text{on } S,\end{aligned}\tag{97}$$

and any  $\bar{E}$  satisfying this will be a satisfactory solution of the problem.

Next consider (96), say the  $\bar{E}$  equation in particular. Taking the divergence of both sides yields

$$\begin{aligned}\nabla \cdot \bar{E} &= -\frac{1}{k^2} \nabla \cdot \nabla^2 \bar{E} \\ &= -\frac{1}{k^2} \nabla \cdot \left[ -\nabla \times \nabla \times \bar{E} + \nabla \nabla \cdot \bar{E} \right] \\ &= -\frac{1}{k^2} \nabla \cdot \left[ \nabla \nabla \cdot \bar{E} \right]\end{aligned}\tag{98}$$

which is not identically zero. Thus, simply finding a solution of the vector wave equation

$$\begin{aligned}\nabla^2 \bar{E} + k^2 \bar{E} &= 0 \\ \bar{n} \times \bar{E} &= 0 \quad \text{on } S\end{aligned}\tag{99}$$

is not sufficient for solving the problem. As a matter of fact solutions of (99) will definitely not satisfy Maxwell's equations unless they also satisfy the zero divergence criterion.

There is a simple example of this. Consider the function  $\phi$  which satisfies the scalar wave equation

$$\begin{aligned}\nabla^2 \phi + k^2 \phi &= 0 \\ \phi &= 0 \quad \text{on } S.\end{aligned}\tag{100}$$

Then the gradient of  $\phi$  will satisfy the vector wave equation (99) and the boundary condition for  $\bar{E}$ . This can be verified by direct substitution in the vector wave equation as follows.

$$\begin{aligned}\nabla^2 (\nabla \phi) + k^2 (\nabla \phi) \\ = -\nabla \times \nabla \times \nabla \phi + \nabla (\nabla \cdot \nabla \phi) + k^2 \nabla \phi \\ = \nabla (\nabla^2 \phi + k^2 \phi) = 0.\end{aligned}$$

But  $\nabla \cdot \nabla \phi$  is just  $-k^2 \phi$  from (100) and thus  $\nabla \phi$  cannot satisfy the zero divergence criterion, as  $\phi$  was tacitly assumed to be a nontrivial solution of (100).

At this point it might not seem worthwhile to deal with the vector wave equation (99) rather than that of (97) which will always give a valid solution to the problem. However, this zero divergence difficulty with the wave equation can be circumvented in a rather obvious way. Assume a solution can be found for the general equation

$$\begin{aligned}\nabla^2 \bar{A} + k^2 \bar{A} &= 0 \\ \bar{n} \times \bar{A} &= 0 \quad \text{on } S\end{aligned}\tag{101}$$

which is not simply the gradient of a scalar function. Then  $\nabla \times \bar{A}$  is also a solution of the same equation. This can be shown by direct substi-

tution as follows:

$$\begin{aligned}
 & \nabla^2 (\nabla \times \bar{A}) + k^2 (\nabla \times \bar{A}) \\
 = & -\nabla \times \nabla \times (\nabla \times \bar{A}) + \nabla \nabla \cdot (\nabla \times \bar{A}) + k^2 (\nabla \times \bar{A}) \\
 = & \nabla \times (-\nabla \times \nabla \times \bar{A} + k^2 \bar{A}) \\
 = & \nabla \times (\nabla^2 \bar{A} - \nabla \nabla \cdot \bar{A} + k^2 \bar{A}) \\
 = & \nabla \times (\nabla^2 \bar{A} + k^2 \bar{A}) \\
 = & 0.
 \end{aligned}$$

Also, if  $\bar{A}$  satisfies the  $\bar{n} \times \bar{A} = 0$  condition on the boundary, then  $\nabla \times \bar{A}$  must satisfy the  $\bar{n} \cdot (\nabla \times \bar{A}) = 0$  boundary condition as  $\nabla \times \bar{A}$  is at right angles to  $\bar{A}$ . Thus  $\nabla \times \bar{A}$  would be a perfectly satisfactory solution for the magnetic field because it has zero divergence and satisfies the wave equation and the proper boundary condition. And further,  $\nabla \times (\nabla \times \bar{A})$  would be a satisfactory solution for the electric field (modified by a constant, of course). Therefore, even though  $\bar{A}$  itself is not a valid solution to the problem, valid ones can be obtained from it by a simple matter of the curl operation. In all of this discussion it is necessary that  $\bar{A}$  not be the gradient of a scalar or  $\nabla \times \bar{A}$  would be identically zero.

In passing, one might think that the same thing could be accomplished by resolving  $\bar{A}$  into its solenoidal and irrotational components<sup>a</sup>. This

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<sup>a</sup>See either Phillips (8) p. 187 or Page (7) p. 45 for a discussion of resolving vector functions into their solenoidal and irrotational components.

is a rather interesting avenue of investigation, but not very fruitful as far as this problem is concerned. It can be shown that if  $\bar{A}$  satisfies the vector wave equation, then both the solenoidal and irrotational components of  $\bar{A}$  will also satisfy it. However, if  $\bar{A}$  satisfies the  $\bar{n} \times \bar{A} = 0$  boundary condition, then the solenoidal component of  $\bar{A}$  will satisfy neither the boundary condition for  $\bar{E}$  or  $\bar{H}$ , and so is not a valid solution for the problem at hand. One cannot help but think that there should be some boundary condition which might be placed on  $\bar{A}$  which would give the appropriate one for the solenoidal component of  $\bar{A}$ , but the solution is not obvious.

By this time one might wonder why all the concern about which differential equation should be solved. Why not just solve (97) rather than the wave equation and be done with it? The answer to this is that the vector wave equation is usually easier to deal with than the other expression. This is especially true when rectangular coordinates are used, as  $\nabla^2$  operating on a vector is simply the Laplacian operator operating on each component of the vector separately. Thus the vector wave equation reduces to three scalar wave equations.

Before proceeding to the problem of the vector Green's function, it might be well to mention some of the properties of solutions of the vector wave equation<sup>a</sup>. As might be expected solutions exist only for discrete values of  $k^2$ , and these values of  $k^2$  determine the resonant frequency

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<sup>a</sup>A detailed treatment of this subject will be found in Slater (12) p. 57.

associated with each of the different modes according to the equation

$$f_r = \frac{k}{2\pi\sqrt{\mu\epsilon}} \quad (102)$$

where  $f_r$  is the resonant frequency in cycles per second. Also, the solutions are orthogonal in the sense that

$$\int_V \bar{E}_m \cdot \bar{E}_n \, dv = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \quad (103)$$

$$\int_V \bar{H}_m \cdot \bar{H}_n \, dv = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n. \end{cases} \quad (104)$$

Degeneracy can occur, but here again linear combinations of the degenerate solutions can be formed which will be orthogonal. All in all, the solutions are quite similar to those of the scalar wave equation, except for being vectors rather than scalars.

#### B. The Green's Function for the Problem

Before discussing the iterative process as applied to the vector wave equation, the matter of a suitable Green's function for the problem will be investigated. The general approach will be the same as before. Consider a vector equation of the form

$$\begin{aligned} \nabla^2 \bar{A} &= \bar{R} \\ \bar{n} \times \bar{A} &= 0 \\ \text{or } \bar{n} \cdot \bar{A} &= 0 \quad \text{on } S \end{aligned} \quad (105)$$

where  $\bar{R}$  is an arbitrary function of  $x$ ,  $y$  and  $z$ . Either of the two boundary conditions may be applied in the discussion which follows, but not both simultaneously, of course. A Green's function must be found such that  $\bar{A}$  may be written as

$$\bar{A}(x, y, z) = \int_V G(x, y, z, \xi, \eta, \varsigma) \bar{R}(\xi, \eta, \varsigma) d\xi d\eta d\varsigma \quad (106)$$

where  $G$  may be thought of for the time being as some sort of operator such as to make the above equation true.

Consider now the possible nature of the operator  $G$ . If one operates on the right side of (106) with  $\nabla^2$  the result must be just  $\bar{R}$ . At first glance it appears that  $G$  could be the scalar potential due to a point charge at  $(\xi, \eta, \varsigma)$  as encountered before, as  $\nabla^2$  operating on it would have the appropriate delta function properties at  $(\xi, \eta, \varsigma)$ . However, this leads to an impossible situation with regard to the boundary conditions on  $\bar{A}$ . As a matter of fact  $G$  cannot be any scalar function. This can be seen by letting  $\bar{R}$  be a unidirectional vector. Then according to (106)  $\bar{A}$  must be in the same direction, and the boundary conditions would be violated. Thus the possibility of using a scalar function for  $G$  is completely ruled out.

Next consider the possibility of using a vector function for  $\bar{G}$ . This being the case,  $\bar{G}$  operating on  $\bar{R}$  must result in a vector, and thus dotting  $\bar{G}$  into  $\bar{R}$  is ruled out. If one tries crossing  $\bar{G}$  into  $\bar{R}$ , the result will always be at right angles to  $\bar{R}$  and this possibility is

also eliminated. To get around this difficulty one could conceivably define a new form of vector product, but this did not prove to be fruitful as far as the author is concerned.

The next step up the ladder as far as complexity of functions is concerned, is a dyadic or second order tensor\*. This appears to be the answer as one can define the components of the dyadic in such a way that (106) will be true. Some of the previous concepts of Green's functions will have to be modified slightly, but most of the basic ideas will remain the same. At this point, in order to keep the notation from becoming unwieldy, let the coordinates of the point  $(x, y, z)$  be denoted by just  $x_i$  and those of  $(\xi, \eta, \zeta)$  by  $\xi_i$ . Then the Green's function will be written as  $\overline{\overline{G}}(x_i, \xi_i)$  where the double bar indicates a "double vector" or dyadic.

Now (106) can be written as

$$\overline{A}(x_i) = \int_V \overline{\overline{G}}(x_i, \xi_i) \cdot \overline{R}(\xi_i) d\xi_i \quad (107)$$

Also, let  $\overline{\overline{G}}$  and  $\overline{R}$  be written in component form as

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\*For purposes of this problem a dyadic and tensor are one and the same thing. The difference is one of notation, and as the dyadic notation fits in better with the standard vector notation than does the tensor notation, the term dyadic will be used throughout.

$$\begin{aligned}
\bar{G} &= g_{11} \bar{i}\bar{i} + g_{12} \bar{i}\bar{j} + g_{13} \bar{i}\bar{k} \\
&+ g_{21} \bar{j}\bar{i} + g_{22} \bar{j}\bar{j} + g_{23} \bar{j}\bar{k} \\
&+ g_{31} \bar{k}\bar{i} + g_{32} \bar{k}\bar{j} + g_{33} \bar{k}\bar{k}
\end{aligned} \tag{108}$$

$$\bar{R} = r_1 \bar{i} + r_2 \bar{j} + r_3 \bar{k} . \tag{109}$$

Then  $\bar{G} \cdot \bar{R}$  would be

$$\begin{aligned}
\bar{G} \cdot \bar{R} &= (g_{11} \bar{i} + g_{21} \bar{j} + g_{31} \bar{k}) r_1 \\
&+ (g_{12} \bar{i} + g_{22} \bar{j} + g_{32} \bar{k}) r_2 \\
&+ (g_{13} \bar{i} + g_{23} \bar{j} + g_{33} \bar{k}) r_3 .
\end{aligned} \tag{110}$$

Now this must be such that

$$\int_V \nabla^2 (\bar{G} \cdot \bar{R}) d\xi_i = \bar{R} ,$$

remembering that  $\nabla^2$  operates with respect to the space coordinates  $\chi_i$  and the integration is with respect to  $\xi_i$ . The quantity  $\nabla^2 (\bar{G} \cdot \bar{R})$  can be written as

$$\nabla^2 (\bar{G} \cdot \bar{R}) = (\nabla^2 \bar{G}_1) r_1 + (\nabla^2 \bar{G}_2) r_2 + (\nabla^2 \bar{G}_3) r_3 \tag{111}$$

where

$$\begin{aligned}
\bar{G}_1 &= g_{11} \bar{i} + g_{21} \bar{j} + g_{31} \bar{k} \\
\bar{G}_2 &= g_{12} \bar{i} + g_{22} \bar{j} + g_{32} \bar{k} \\
\bar{G}_3 &= g_{13} \bar{i} + g_{23} \bar{j} + g_{33} \bar{k} .
\end{aligned} \tag{112}$$



In order for  $\bar{G}$  to be a suitable Green's function each of the terms in parentheses of (111) must have magnitude properties of a delta function and vector sense in the  $x$ ,  $y$  and  $z$  directions respectively for the three terms. This being the case the integral of (111) would then yield  $\bar{R}$  as it should. The expressions for the vector potential due to point currents at  $\xi_i$  in each of the  $x$ ,  $y$  and  $z$  directions are just the type of functions needed here. Thus in the neighborhood of  $\xi_i$

$$\bar{G}_1 \approx -\frac{i}{4\pi r}, \quad \bar{G}_2 \approx -\frac{j}{4\pi r}, \quad \bar{G}_3 \approx -\frac{k}{4\pi r} \quad (113)$$

where  $r$  is the distance from  $\xi_i$  to  $x_i$ .

However, while these vector functions have the appropriate properties in the neighborhood of  $\xi_i$ , they do not at the boundary. Thus, each must have added to it another vector function satisfying  $\nabla^2 \bar{A} = 0$  within  $S$  and such that  $\bar{G}_1$ ,  $\bar{G}_2$ , and  $\bar{G}_3$  will satisfy the appropriate boundary condition. That is,  $\bar{G}_1$ ,  $\bar{G}_2$ , and  $\bar{G}_3$  must each satisfy the boundary condition  $\bar{n} \cdot \bar{A} = 0$  or  $\bar{n} \times \bar{A} = 0$ , depending on which case is being considered. These added functions might be thought of as due to induced currents on the boundary, although the magnetic analogy requires a little imagination. The problem is very similar to that encountered in the electrostatic case where induced charges on the boundary are necessary in order to satisfy the boundary condition. In general the problem of finding the required current distribution is not an easy one. However, that such a solution should exist, for  $\bar{G}_1$  say, seems reasonable, as each of the separate components of  $\bar{G}_1$  is just a Dirichlet problem in itself<sup>a</sup>.

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<sup>a</sup>See Phillips (8) p. 165 for a discussion of the Dirichlet problem.

It might be pointed out in passing that the term Green's function as used here means something altogether different than the term as used by Stratton (13). Stratton considers the vector Green's function to be any suitable vector function which, along with the vector form of Green's theorem, will enable one to integrate directly the vector equation

$$\nabla \times \nabla \times \bar{A} = \mu \bar{J}. \quad (114)$$

Here the term Green's function is used in the same sense as in Schwinger's report on the theory of obstacles in wave guides and cavities (11), although it is not the same identical function. The author is, however, indebted to Schwinger for the idea of using a dyadic Green's function for the vector problem.

### C. Method of Successive Approximations Applied to Vector Wave Equation

In the resonant cavity problem one may solve for either the electric or magnetic field, and then obtain the other by taking the curl of the one for which a solution has already been found. The electric field has been chosen to demonstrate the method, but as far as the method of successive approximations is concerned there is no particular advantage in working with one over the other. The electric field must satisfy the equation

$$\begin{aligned} \nabla^2 \bar{E} + k^2 \bar{E} &= 0 \\ \bar{n} \times \bar{E} &= 0 \quad \text{on } S \end{aligned} \quad (115)$$

and the zero divergence criterion. However, as has been shown previously, the zero divergence criterion is not a serious matter, as the correct magnetic field will still be given by  $\nabla \times \bar{E}$  even if  $\bar{E}$  does not have zero divergence.

As before, the iterative process begins by assuming some vector function  $\bar{E}^0$  which satisfies the boundary conditions, and then obtaining from it a new approximation  $\bar{E}'$  from the equation

$$\nabla^2 \bar{E}' = -k^2 \bar{E}^0 \quad (116)$$

Then  $\bar{E}'$  is used to obtain  $\bar{E}^2$ , and so forth.

In order to show that this process will converge to a solution of the original problem, certain vector functions must be expanded in terms of a suitable set of orthogonal functions. No proof will be given here for the validity of such an expansion, and Slater (12) may be referred to for a more thorough treatment of the subject. Briefly, Slater shows that the eigenfunctions of both the  $\bar{E}$  and  $\bar{H}$  field problems form infinite sets of orthogonal functions, either of which may be used for expanding an arbitrary solenoid function satisfying certain not too stringent continuity requirements. The only advantage of using one set over the other would lie in the rate of convergence of the expansion. However, if the function to be expanded has both solenoidal and irrotational components, then another infinite set of orthogonal irrotational functions must be added to the solenoidal set in order to have a complete set for the expansion. Such an irrotational set can be obtained by taking the gradient of each of the eigenfunctions of the scalar wave equation problem

$$\begin{aligned}\nabla^2 \phi + k^2 \phi &= 0 \\ \phi &= 0 \quad \text{on } S.\end{aligned}\tag{117}$$

These vector functions are orthogonal not only with respect to themselves, but also those of the solenoidal set.

Returning to the original problem, the solution of (116) can be written in the form

$$\bar{E}'(\chi_i) = -\lambda' \int_V \bar{G}(\chi_i, \xi_i) \cdot \bar{E}^o(\xi_i) d\xi_i \tag{118}$$

where  $k^2$  has been replaced with the parameter  $\lambda$  in order to avoid confusion about superscripts. Now, at this point it will be convenient to group the terms of  $\bar{G}$  in a little different way than was done in the previous section. There, each column of the  $g_{mn}$  matrix was thought of as constituting a vector and these were denoted  $\bar{G}_1$ ,  $\bar{G}_2$  and  $\bar{G}_3$ . Here the terms will be grouped in rows rather than columns, i.e.  $\bar{G}$  will be written as

$$\bar{G} = \bar{i} \bar{X} + \bar{j} \bar{Y} + \bar{k} \bar{Z} \tag{119}$$

where

$$\begin{aligned}\bar{X} &= g_{11} \bar{i} + g_{12} \bar{j} + g_{13} \bar{k} \\ \bar{Y} &= g_{21} \bar{i} + g_{22} \bar{j} + g_{23} \bar{k} \\ \bar{Z} &= g_{31} \bar{i} + g_{32} \bar{j} + g_{33} \bar{k}.\end{aligned}\tag{120}$$

Then

$$\bar{G} \cdot \bar{E}^o = \bar{i} (\bar{X} \cdot \bar{E}^o) + \bar{j} (\bar{Y} \cdot \bar{E}^o) + \bar{k} (\bar{Z} \cdot \bar{E}^o). \quad (121)$$

Now, the integral of this vector function is just the integral of each term separately, so at this point each of the terms in parentheses will be expanded by means of the orthogonal functions just mentioned. Before doing this, however, it might be pointed out that the vector components  $\bar{X}$ ,  $\bar{Y}$  and  $\bar{Z}$  of the dyadic  $\bar{G}$  do not, in general, satisfy the boundary conditions of the problem, nor do they have any particular physical interpretation. However, this does not prevent an expansion in terms of functions which do satisfy the boundary conditions. There exists an analogy in the case of an ordinary Fourier series expansion of an arbitrary function within the interval from 0 to  $\pi$ . The arbitrary function does not have to satisfy the same boundary conditions as  $\sin nx$  in order for a valid expansion in terms of  $\sin nx$  to exist within the interval.

The orthogonal sets of functions to be used here will be defined as follows. Let  $\bar{\Psi}_1$ ,  $\bar{\Psi}_2$ , ... be the solenoidal eigenfunctions associated with the problem

$$\begin{aligned} \nabla^2 \bar{\Psi} + \lambda \bar{\Psi} &= 0 \\ \bar{n} \times \bar{\Psi} &= 0 \quad \text{on } S, \end{aligned} \quad (122)$$

and  $\bar{\Phi}_1$ ,  $\bar{\Phi}_2$ , ... the irrotational ones obtained by taking the gradient of the eigenfunctions of the scalar equation

$$\begin{aligned}\nabla^2 \phi + \Lambda \phi &= 0 \\ \phi &= 0 \quad \text{on } S.\end{aligned}\tag{123}$$

In general the eigenvalues of (122) are different from those of (123). Then  $\bar{E}^0$ ,  $\bar{X}$ ,  $\bar{Y}$ , and  $\bar{Z}$  can be written in expanded form as

$$\begin{aligned}\bar{E}^0(\xi_i) &= e_1 \bar{\Psi}_1(\xi_i) + e_2 \bar{\Psi}_2(\xi_i) + \dots \\ &\quad + e'_1 \bar{\Phi}_1(\xi_i) + e'_2 \bar{\Phi}_2(\xi_i) + \dots \\ \bar{X}(x_i, \xi_i) &= \chi_1(x_i) \bar{\Psi}_1(\xi_i) + \chi_2(x_i) \bar{\Psi}_2(\xi_i) + \dots \\ &\quad + \chi'_1(x_i) \bar{\Phi}_1(\xi_i) + \chi'_2(x_i) \bar{\Phi}_2(\xi_i) + \dots \\ \bar{Y}(x_i, \xi_i) &= y_1(x_i) \bar{\Psi}_1(\xi_i) + y_2(x_i) \bar{\Psi}_2(\xi_i) + \dots \\ &\quad + y'_1(x_i) \bar{\Phi}_1(\xi_i) + y'_2(x_i) \bar{\Phi}_2(\xi_i) + \dots \\ \bar{Z}(x_i, \xi_i) &= z_1(x_i) \bar{\Psi}_1(\xi_i) + z_2(x_i) \bar{\Psi}_2(\xi_i) + \dots \\ &\quad + z'_1(x_i) \bar{\Phi}_1(\xi_i) + z'_2(x_i) \bar{\Phi}_2(\xi_i) + \dots\end{aligned}\tag{124}$$

where the coefficients are given by

$$\begin{aligned}e_n &= \int_V \bar{E}^0(\xi_i) \cdot \bar{\Psi}_n(\xi_i) d\xi_i \\ e'_n &= \int_V \bar{E}^0(\xi_i) \cdot \bar{\Phi}_n(\xi_i) d\xi_i \\ &\dots \dots \dots\end{aligned}\tag{125}$$

Then

$$\begin{aligned}
 \int_V \bar{G} \cdot \bar{E}^0 d\xi_i &= \bar{i} \int_V \bar{X} \cdot \bar{E}^0 d\xi_i + \bar{j} \int_V \bar{Y} \cdot \bar{E}^0 d\xi_i + \bar{k} \int_V \bar{Z} \cdot \bar{E}^0 d\xi_i \\
 &= \bar{i} \left[ e_1 \chi_1(x_i) + e_2 \chi_2(x_i) + \dots e'_1 \chi'_1(x_i) + e'_2 \chi'_2(x_i) + \dots \right] \\
 &\quad + \bar{j} \left[ e_1 y_1(x_i) + e_2 y_2(x_i) + \dots e'_1 y'_1(x_i) + e'_2 y'_2(x_i) + \dots \right] \\
 &\quad + \bar{k} \left[ e_1 z_1(x_i) + e_2 z_2(x_i) + \dots e'_1 z'_1(x_i) + e'_2 z'_2(x_i) + \dots \right].
 \end{aligned} \tag{126}$$

But  $\bar{\Psi}_n$  is a solution of

$$\nabla^2 \bar{\Psi}_n + \lambda_n \bar{\Psi}_n = 0, \tag{127}$$

and therefore  $\bar{\Psi}_n$  can be written as

$$\begin{aligned}
 \bar{\Psi}_n &= -\lambda_n \int_V \bar{G} \cdot \bar{\Psi}_n d\xi_i \\
 &= -\lambda_n \left[ \bar{i} \int_V \bar{X} \cdot \bar{\Psi}_n d\xi_i + \bar{j} \int_V \bar{Y} \cdot \bar{\Psi}_n d\xi_i + \bar{k} \int_V \bar{Z} \cdot \bar{\Psi}_n d\xi_i \right].
 \end{aligned} \tag{128}$$

Now, substituting from (125) into (128) gives

$$\bar{\Psi}_n = -\lambda_n \left[ \bar{i} \chi_n + \bar{j} y_n + \bar{k} z_n \right]. \tag{129}$$

Similarly,  $\bar{\Phi}_n$  can be written as

$$\bar{\Phi}_n = -\Lambda_n \left[ \bar{i} x'_n + \bar{j} y'_n + \bar{k} z'_n \right]. \quad (130)$$

Thus, grouping together the columns of (128) and using (129) and (130) leads to

$$\int_V \bar{G} \cdot \bar{E}^o d\xi_i = - \left[ \frac{e_1}{\lambda_1} \bar{\Psi}_1 + \frac{e_2}{\lambda_2} \bar{\Psi}_2 + \dots \right. \\ \left. + \frac{e'_1}{\Lambda_1} \bar{\Phi}_1 + \frac{e'_2}{\Lambda_2} \bar{\Phi}_2 + \dots \right], \quad (131)$$

and finally, the expression for  $\bar{E}'$  is

$$\bar{E}' = -\lambda' \int_V \bar{G} \cdot \bar{E}^o d\xi_i \\ = \frac{\lambda'}{\lambda_1} \left[ e_1 \bar{\Psi}_1 + \left( \frac{\lambda_1}{\lambda_2} \right) e_2 \bar{\Psi}_2 + \dots \right. \\ \left. + \left( \frac{\lambda_1}{\Lambda_1} \right) e'_1 \bar{\Phi}_1 + \left( \frac{\lambda_1}{\Lambda_2} \right) e'_2 \bar{\Phi}_2 + \dots \right]. \quad (132)$$

The line of reasoning from here on is the same as in the previous case of the scalar wave equation. The process will converge to the eigenfunction with the lowest eigenvalue. Any higher order mode can be found by first finding all of the modes below the one of interest and then eliminating them from the initial  $\bar{E}^o$  function. Then the iterative process



would converge to the mode of interest. It is interesting to note that if  $\Lambda_1$  is less than  $\lambda_1$ , the eigenfunction  $\bar{\Phi}_1$  will be obtained. This is of no value in the resonant cavity problem as it is irrotational. However, if  $\bar{E}^o$  is chosen such that it is solenoidal, then all of the  $\bar{\Phi}_n$  terms vanish from (132) and a solenoidal solution is assured. Thus, there would be quite an advantage in initially choosing a solenoid function for  $\bar{E}^o$ .

Again one cannot help but be impressed with the tremendous amount of work which would be involved if one were to try to use this technique to obtain numerical results for a specific problem. However, even though the method will never be used as such, it is nice to know that it does converge, and some of the basic ideas will be carried over into a modification of this method which is taken up in the next section.

V. MODIFICATION OF METHOD SUCH AS TO AVOID  
THE GREEN'S FUNCTION PROBLEM

A. Direct Integration Approach

1. Remarks on solution of Poisson's equation

In the previous sections a great deal of use has been made of the Green's function concept. This was primarily a matter of convenience in proving some generalities regarding convergence of the method and not a matter of absolute necessity. A direct integration approach may be used. For example, consider the familiar differential system

$$\begin{aligned} \frac{d^2 y}{dx^2} &= r(x) \\ y(a) &= y(b) = 0. \end{aligned} \tag{133}$$

Instead of writing the solution in terms of the Green's function as

$$y = \int_a^b G(x, \xi) r(\xi) d\xi \tag{134}$$

one could integrate (133) directly and use the boundary conditions to determine the constants of integration. The resulting solution should, of course, be the same as that obtained from (134).

A somewhat analogous approach exists for the scalar wave equation problem. It will be recalled that each of the iterative steps of the

method just described involves the solution of an equation of the form

$$\begin{aligned} \nabla^2 \phi &= -\rho \\ \left. \begin{aligned} \phi &= 0 \\ \text{or } \frac{\partial \phi}{\partial n} &= 0 \end{aligned} \right\} \text{ on } S \end{aligned} \quad (135)$$

where  $\rho$  is a known function of position. This will be recognized as Poisson's equation, and if  $\nabla^2$  is considered to be a three dimensional operator for the time being, the solution may be written in the form<sup>a</sup>

$$\phi(x, y, z) = \frac{1}{4\pi} \int_V \frac{\rho}{r} dv + \frac{1}{4\pi} \int_S \left( \frac{\nabla \phi}{r} - \phi \nabla \frac{1}{r} \right) \cdot d\bar{S} \quad (136)$$

where  $r$  is the distance between the point  $(x, y, z)$  and the differential element of integration. This solution is obtained with the aid of Green's formula, and the process is sometimes referred to as direct integration of Poisson's equation.

Each of the terms in (136) has an electrostatic interpretation. The first term is independent of the boundary conditions and might be thought of as the potential at point  $(x, y, z)$  due to a charge distribution of density  $\rho$  within the region. The second term represents an induced

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<sup>a</sup>See Lass (4) p. 155.

effect on the boundary, and it must be such that  $\phi$  will satisfy the boundary conditions. If either  $\nabla\phi$  or  $\phi$  is specified on  $S$ , this is sufficient (along with  $\rho$ ) to determine  $\phi$  within  $S$ <sup>a</sup>. For example, if  $\phi$  is zero on  $S$ , then

$$\phi = \frac{1}{4\pi} \int_V \frac{\rho}{r} dv + \frac{1}{4\pi} \int_S \frac{\nabla\phi}{r} \cdot d\bar{s}, \quad (137)$$

or if  $\nabla\phi \cdot d\bar{s}$  is zero on  $S$ , then  $\phi$  is given by

$$\phi = \frac{1}{4\pi} \int_V \frac{\rho}{r} dv + \frac{1}{4\pi} \int_S \left(-\phi \nabla \frac{1}{r}\right) \cdot d\bar{s}. \quad (138)$$

The surface integral term of (137) will be recognized as the potential due to induced charge on a perfectly conducting boundary, and the corresponding term in equation (138) can be thought of as the potential due to a double layer of charge on the boundary.

In both cases, however, it should be noted that  $\phi$  has not been written explicitly as a function of  $\rho$ . Rather,  $\phi$  has merely been written in integral equation form, because either  $\phi$  or  $\nabla\phi$  is involved in the surface integral term. Thus, strictly speaking, the differential equation has not been solved but just transformed to integral equation form.

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<sup>a</sup>This follows directly from the first and second boundary value problems of potential theory (Dirichlet and Neumann problems). See Phillips (8) pp. 165-174.

## 2. Two-dimensional case

The solution of the two-dimensional form of Poisson's equation is not quite as well known as the three dimensional one, so it will be derived. It is the form directly applicable to the wave guide problems of interest. Consider Green's formula as applied to the volume between the small cylindrical surface  $\Sigma'$  and the outer surface  $\Sigma$  and of unit length as shown in Fig. 3. Green's formula states that

$$\int_{V'} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \int_{S'} (\phi \nabla \psi - \psi \nabla \phi) \cdot d\bar{s} \quad (139)$$

where  $V'$  is the volume between  $\Sigma$  and  $\Sigma'$  and  $S'$  is the surface enclosing this volume.

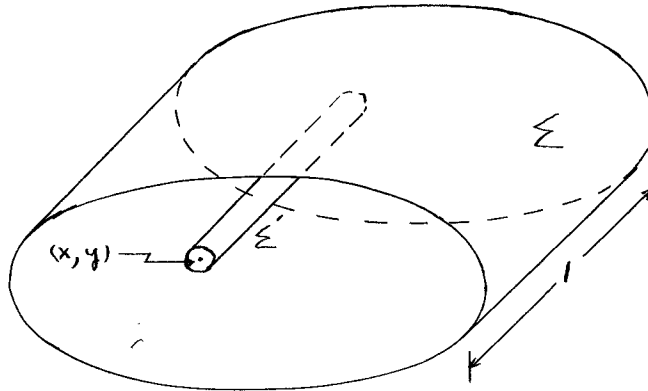


Fig. 3 Section of Guide of Unit Length

Now let

$\phi$  = Potential function of interest

$\psi$  =  $-\ln r$

where  $r$  is the distance from the line at  $(x, y)$  to any point within  $V'$ .

Then if

$$\nabla^2 \phi = -\rho$$

and

$$\nabla^2 \psi = 0,$$

Green's formula becomes

$$\begin{aligned} \int_{V'} -\rho \ln r \, dv &= \int_{\Sigma} [\phi \nabla(-\ln r) + \ln r \nabla \phi] \cdot \bar{ds} \\ &+ \int_{\Sigma'} [\phi \nabla(-\ln r) + \ln r \nabla \phi] \cdot \bar{ds}. \end{aligned} \quad (140)$$

Note that the integral over the end pieces is zero because there is no variation in the longitudinal direction. The latter term of (140) can be evaluated if the radius of the surface  $\Sigma'$  is assumed small. Letting the radius be  $\delta$ , and remembering  $\bar{ds}$  is directed inward in this case, the terms of the integral reduce to

$$\int_{\Sigma'} \phi \nabla(-\ln r) \cdot \bar{ds} = \int_{\Sigma'} \phi \left(-\frac{1}{r}\right) \bar{1}_r \cdot \bar{ds} \approx \phi \frac{1}{\delta} 2\pi \delta = 2\pi \phi$$

$$\int_{\Sigma'} \ln r \nabla \phi \cdot \bar{ds} \leq |\nabla \phi_{\max}| |\ln \delta| |2\pi \delta| \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

One has only to write  $(\ln \delta)(2\pi \delta)$  in the form  $\frac{\ln \delta}{\frac{1}{2\pi \delta}}$  and differ-

entiate numerator and denominator to see that this will approach zero as

$\delta$  approaches zero. Thus  $\phi$  can now be written as

$$\phi(x, y) = \frac{1}{2\pi} \int_{V'} -\rho \ln r \, dv - \frac{1}{2\pi} \int_{\Sigma} [\phi \nabla(-\ln r) + \ln r \nabla \phi] \cdot \bar{ds}. \quad (141)$$

As the section considered is of unit length and there is no variation in the  $z$  direction, the volume and surface integrals of (141) can be replaced with surface and line integrals respectively. The resulting expression for  $\phi$  is

$$\phi(x,y) = -\frac{1}{2\pi} \left[ \int_S \rho \ln r \, ds + \int_C (\nabla\phi \ln r - \phi \nabla \ln r) \cdot d\bar{l} \right] \quad (142)$$

where  $d\bar{l}$  indicates the differential element of contour  $C$  with a vector sense normal to  $C$  and directed outward, and  $S$  is the cross-sectional area of the guide. Thus, (142) is the two dimensional counterpart of (136), and the same electrostatic interpretation of each of the terms applies here also.

### 3. Modification of the iterative procedure

Returning to the original problem, the iterative process for solving the wave guide problem involves solving the equation

$$\nabla^2 \phi^n = -\lambda^n \phi^{n-1} \quad (143)$$

where  $\nabla^2$  is the two-dimensional Laplacian operator. Using (142) the expression for  $\phi^n$  can be written as

$$\phi^n = -\frac{1}{2\pi} \left[ \lambda^n \int_S \phi^{n-1} \ln r \, ds + \int_C (\nabla\phi^n \ln r - \phi^n \nabla \ln r) \cdot d\bar{l} \right] \quad (144)$$

It has been shown previously that the  $\phi^n$  obtained by this procedure will

converge to the lowest order eigenfunction contained in  $\phi^0$ . However,  $\phi^n$  cannot be obtained directly from (144) because it appears within the line integral term. For large  $n$ , though,  $\phi^n \approx \phi^{n-1}$  and  $\lambda^n \approx \lambda^{n-1}$ . This leads one to wonder if it might not be possible to use  $\phi^{n-1}$  in the line integral term rather than  $\phi^n$ . After all,  $\phi^n$  is only an approximation to the eigenfunction of the problem. Replacing  $\phi^n$  with  $\phi^{n-1}$  in (144) will not lead to the same  $\phi^n$  as given in the original equation, but the difference should become smaller and smaller as  $n$  becomes large, providing, of course, that the procedure converges. As the two methods are similar for large  $n$ , the same arguments regarding speed of convergence and degeneracy previously discussed should carry over directly.

The advantage of this modification should be obvious. One has only to evaluate integrals of known functions this way, whereas in the original scheme, an integral equation had to be solved with each step. Showing that this modified procedure will converge is not a simple matter, and nothing general was developed along this line. As a poor substitute the method was tried on a rectangular wave guide example and it seemed to converge as expected. This example, along with the details of the method, follows in the next section.

### B. Rectangular Wave Guide Example

Examples which are not completely trivial, and yet simple enough to work with just the aid of a slide rule and desk calculator, are not very



numerous. The problem finally chosen was that of the TM mode case for a square wave guide. This problem has enough symmetry to be workable with a reasonable amount of effort, and yet is extreme in the sense that the bounding curve has sharp corners. These advantages, along with the fact that an exact solution is available for comparison, make it an ideal example for experimenting with the modified method.

For the TM case  $\phi$  (which is being used in place of  $E_z$ ) must be zero on the boundary. Thus (144) reduces to

$$\begin{aligned}\phi^n &= -\frac{1}{2\pi} \left[ \lambda^n \int_S \phi^{n-1} \ln r \, ds + \int_C (\nabla \phi^n \ln r) \cdot d\bar{l} \right] \\ &= -\frac{\lambda^n}{2\pi} \left[ \int_S \phi^{n-1} \ln r \, ds - \int_C \frac{\partial}{\partial n} \left( \frac{\phi^n}{\lambda^n} \right) \ln r \, |dl| \right] \quad (145)\end{aligned}$$

where  $\frac{\partial}{\partial n}$  refers to the normal derivative directed inward. The terms

have been grouped in such a way that  $\lambda^n$  can be factored out in front of the whole expression. At this point  $\frac{\phi^n}{\lambda^n}$  in the line integral term of (145) will be replaced with  $\frac{\phi^{n-1}}{\lambda^{n-1}}$ , giving for the final recursion formula

$$\phi^n \approx -\frac{\lambda^n}{2\pi} \left[ \int_S \phi^{n-1} \ln r \, ds - \int_C \frac{\partial}{\partial n} \left( \frac{\phi^{n-1}}{\lambda^{n-1}} \right) \ln r \, |dl| \right] \quad (146)$$

where  $\lambda^n$  will be chosen such as to normalize the resulting  $\phi^n$  as before.

For computational purposes the region  $S$  was divided into 100 incremental squares. This seemed to be a reasonable compromise between the desired accuracy and the amount of work involved. The purpose of the

example was not to find a precise solution to the problem, but rather to get some idea as to whether or not the process would converge. As a matter of convenience the incremental area was chosen to be unity and thus making the whole square 10 by 10 units. Due to symmetry the value of  $\phi^n$  had to be determined only in the 15 squares shown numbered in Fig. 4. For identification purposes each square was numbered beginning with one in the upper left corner.

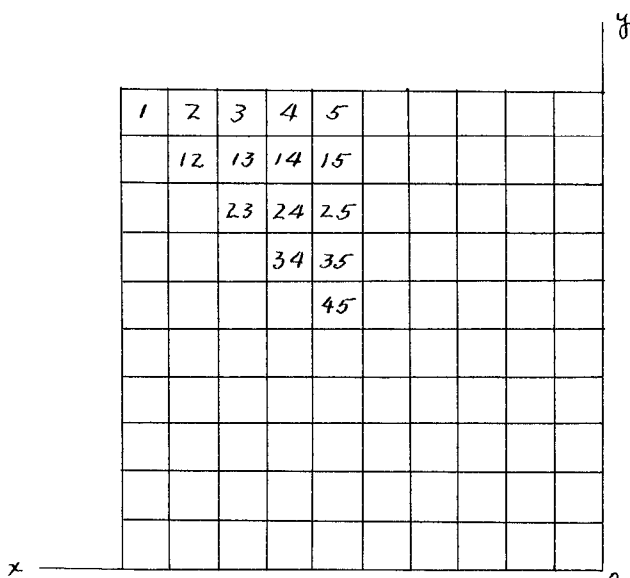


Fig. 4 Numbering Scheme for Incremental Squares

In order to simplify the calculations the function was assumed to be constant throughout each interval, and center to center distances were used in the  $\ln r$  terms. This is the equivalent of saying that the effect of the distributed charge within the boundary can be replaced with 100 fine lines of charge, one at the center of each unit square, and that the in-

duced charge can be replaced with forty fine lines distributed at unit intervals along the bounding curve  $C$ . There were two exceptions to this rule, however. In computing the component of potential at a point due to charge within that same region, obviously another approach must be used. It was decided to consider the equivalent effect of a circular region of unit area which can be computed readily. Also, in finding the effect of the induced charge on the boundary on the potential in an adjacent square, 0.6 unit was thought to be a better approximation than 0.5 for the equivalent distance from one side to the center of the square.

A pyramid shaped function with its peak at the center of the square was chosen for the initial  $\phi^0$  function. This was a convenient function to begin with because it can be normalized easily and has constant slope everywhere along the edge which makes the induced effects easy to compute.

In addition a value for  $\lambda^0$  had to be assumed before the  $\frac{\partial}{\partial n} \left( \frac{\phi^0}{\lambda^0} \right)$  term could be evaluated. This involved a little thought. If one expects the resultant potential function  $\phi'$  to be approximately zero on the boundary, there should be about as much induced charge on the boundary as distributed charge within the region. Thus one criterion for choosing  $\lambda^0$  would be to make it such that the sum of all the  $\phi^0 ds$  terms will equal the sum of the

$\frac{\partial}{\partial n} \left( \frac{\phi^0}{\lambda^0} \right) |dl|$  terms. This criterion would be worthless, however, in

evaluating some of the higher order modes where the total induced charge is zero.

Another way to choose  $\lambda^0$  would be to use the approximate eigenvalue

corresponding to  $\phi^0$  as given by the Ritz method. It can be shown that the present problem can be formulated in a variational way, and one of the results is that the eigenvalue  $\lambda$  is the minimum of the function

$$\int_S \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] ds, \text{ subject to the normalizing constraint on } \phi \text{ that}^a$$

$$\int_S \phi^2 ds = 1.$$

Thus an approximate value of  $\lambda$  may be associated with each approximate eigenfunction in accordance with this surface integral. For example the  $\lambda^0$  corresponding to  $\phi^0$  would be given by

$$\lambda^0 = \int_S \left[ \left( \frac{\partial \phi^0}{\partial x} \right)^2 + \left( \frac{\partial \phi^0}{\partial y} \right)^2 \right] ds \quad (147)$$

and for the particular  $\phi^0$  chosen for this example  $\lambda^0$  works out to be 0.240. This value also makes the total induced charge approximately equal to the charge within  $S$ , so it was used for the initial value of  $\lambda$ .

The correct normalized solution for the problem is

$$\phi(x, y) = 0.2 \sin \frac{\pi x}{10} \sin \frac{\pi y}{10}, \quad (148)$$

and the corresponding eigenvalue is  $\frac{\pi^2}{50}$  or about 0.197. This solution may be verified by referring to any standard text on wave guide theory.

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<sup>a</sup>See Weinstock (15) p. 164.

The iterative process was carried through two steps, and the results are tabulated in Table 2. The correct values listed in the table were computed from (148). Only the values for the 15 squares noted in Fig. 4 are given, as the others can be obtained from the symmetry of the problem.

Table 2. Results of Two Iterative Steps

Coordinate*	$\phi^0$	$\phi^1$	$\phi^2$	Correct Value
1	.024	-.004	.000	.005
2	.024	.015	.013	.014
3	.024	.027	.026	.022
4	.024	.040	.035	.028
5	.024	.048	.038	.031
12	.074	.041	.041	.041
13	.074	.064	.065	.064
14	.074	.084	.083	.081
15	.074	.093	.096	.090
23	.123	.101	.101	.100
24	.123	.124	.122	.126
25	.123	.138	.140	.140
34	.172	.156	.157	.159
35	.172	.172	.174	.176
45	.222	.191	.192	.195

\*Numbering scheme for incremental squares is given in Fig. 4.

Table 3 shows the approximate eigenvalues obtained with each step and the correct value for comparison. Note that  $\lambda^2$  is within about 2.5 per cent of the correct value, which is, perhaps, about as close as might be expected for the rather coarse incremental method used.

Table 3. Successive Approximations for  $\lambda$ 


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$\lambda^0$ . . . . .	0.240
$\lambda^1$ . . . . .	0.184
$\lambda^2$ . . . . .	0.192
$\lambda$ (correct) . . . . .	0.197

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The final results for  $\phi^2$  are also shown graphically in Fig. 5 along with the initial  $\phi^0$  curves. The solid curves show the value of the function  $\phi^2$  for five cross-sectional cuts, one through each tier beginning with the outside row of squares. The dotted curves represent the corresponding initial  $\phi^0$  function. In both cases curves are plotted only for one quadrant of the square because of the symmetry. It can be seen that after two iterative steps the curves are beginning to look like sine functions.

### C. Conclusions and Extension to Cavity Problem

One cannot conclude anything very general just on the basis of the results of one example. However, it certainly appears that the technique could be applied to other similar problems with the homogeneous boundary condition on  $\phi$ . Nor does there appear to be any reason why the method should not work just as well for the zero normal derivative boundary con-

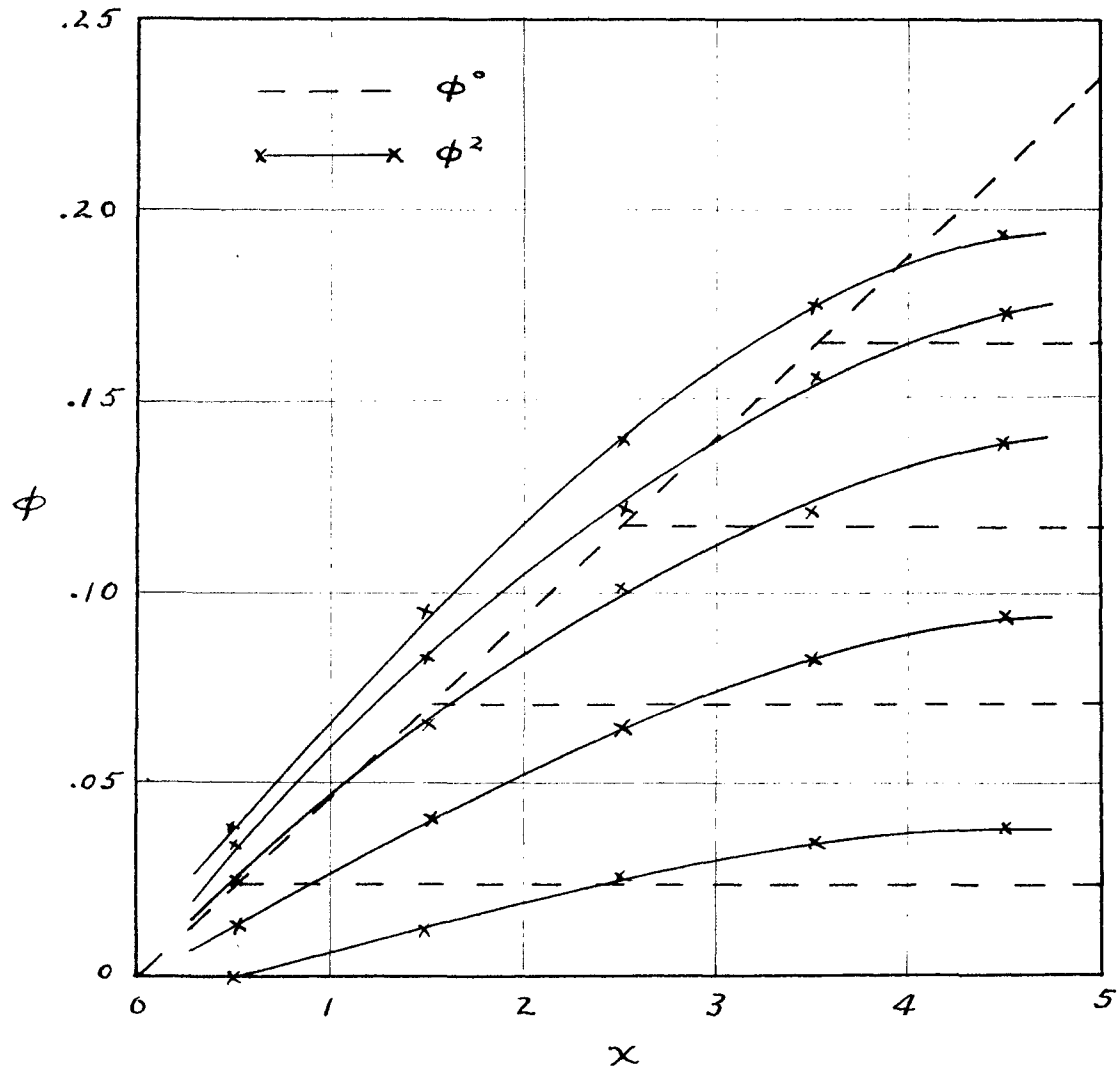


Fig. 5 Cross Sections of  $\phi^0$  and  $\phi^2$ .

dition case. Using the modified version is not quite as routine as the original scheme though, as one must use a little judgement in choosing the proper combination of slope and  $\lambda$  in the first few steps in order to make the induced and direct charge terms sum to zero. For example in the preceding problem the slope of  $\phi'$  was difficult to determine because of the coarse resolution, and so the zero total charge criterion was used to help determine the slope of  $\phi'$ .

Finding the higher order solutions might prove to be difficult using this modified approach. It will be recalled that in the original scheme, absence of any particular mode in the initial  $\phi^0$  also insured the absence of that mode in each of the successive approximations. Thus one could obtain the second order solution, say, by using an initial function not containing the fundamental. However, this is not so with the modified version of the method. The fundamental might show up in  $\phi'$  even though not present in  $\phi^0$  because of the approximations made in the equation for  $\phi'$ . It would appear that the only way to insure convergence on the second order solution would be to remove the fundamental component from each successive approximation. This could be done, in principle at least, if the fundamental were known, but it would involve a considerable amount of work.

With regard to extending the method to the resonant cavity problem, it can be shown that the vector equation

$$\nabla \times \nabla \times \bar{A} = \mu \bar{J} \quad (149)$$

can be integrated (in the same sense of the word as in the case of Poisson's



equation), and the result is<sup>a</sup>

$$\bar{A}(x, y, z) = \frac{1}{4\pi} \left[ \int_V \frac{\mu \bar{J}}{r} dv - \int_S \frac{\bar{n} \times (\nabla \times \bar{A})}{r} ds - \int_S (\bar{n} \times \bar{A}) \times \nabla \left( \frac{1}{r} \right) ds - \int_S (\bar{n} \cdot \bar{A}) \nabla \left( \frac{1}{r} \right) ds \right] \quad (150)$$

Here again  $\bar{A}$  has not been solved for explicitly but is merely given in integral equation form. In the cavity problem the recursion formula encountered is\*

$$\nabla \times \nabla \times \bar{E}^n = \lambda^n \bar{E}^{n-1}, \quad (151)$$

and the solution for  $\bar{E}^n$  may be written using (150). Here, as before,  $\bar{E}^n \approx \bar{E}^{n-1}$  and  $\lambda^n \approx \lambda^{n-1}$  for large  $n$ . Thus one might think that there is a possibility of doing something similar here to that of the previous case. No attempt was made to work any examples along this line, so this is merely suggested as a possibility.

<sup>a</sup>See Stratton (13) p. 250.

\*As long as  $\bar{E}^n$  is solenoidal,  $\nabla \times \nabla \times \bar{E}^n = -\nabla^2 \bar{E}^n$ .

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## VII. ACKNOWLEDGEMENTS

The author wishes to thank his major professor Dr. L. W. Von Tersch for his guidance in this investigation and Dr. L. J. Laslett of the Physics Department for his help in obtaining some literature pertinent to the subject.